I, §1. DEFINITIONS

Let K be a subset of the complex numbers **C**. We shall say that K is a **field** if it satisfies the following conditions:

(a) If x, y are elements of K, then $x + y$ and xy are also elements of K.
(b) If $x \in K$, then $-x$ is also an element of K. If furthermore $x \neq 0$, then x^{-1} is an element of K.
(c) The elements 0 and 1 are elements of K.

We observe that both **R** and **C** are fields.

Let us denote by **Q** the set of rational numbers, i.e. the set of all fractions m/n, where m, n are integers, and $n \neq 0$. Then it is easily verified that **Q** is a field.

Let **Z** denote the set of all integers. Then **Z** is not a field, because condition (b) above is not satisfied. Indeed, if n is an integer $\neq 0$, then $n^{-1} = 1/n$ is not an integer (except in the trivial case that $n = 1$ or $n = -1$). For instance $\frac{1}{2}$ is not an integer.

The essential thing about a field is that it is a set of elements which can be added and multiplied, in such a way that additon and multiplication satisfy the ordinary rules of arithmetic, and in such a way that one can divide by non-zero elements. It is possible to axiomatize the notion further, but we shall do so only later, to avoid abstract discussions which become obvious anyhow when the reader has acquired the necessary mathematical maturity. Taking into account this possible generalization, we should say that a field as we defined it above is a field of (complex) numbers. However, we shall call such fields simply fields.

The reader may restrict attention to the fields of real and complex numbers for the entire linear algebra. Since, however, it is necessary to deal with each one of these fields, we are forced to choose a neutral letter K.

Let K, L be fields, and suppose that K is contained in L (i.e. that K is a subset of L). Then we shall say that K is a **subfield** of L. Thus every one of the fields which we are considering is a subfield of the complex numbers. In particular, we can say that **R** is a subfield of **C**, and **Q** is a subfield of **R**.

Let K be a field. Elements of K will also be called **numbers** (without specification) if the reference to K is made clear by the context, or they will be called **scalars**.

A **vector space** V **over the field** K is a set of objects which can be added and multiplied by elements of K, in such a way that the sum of two elements of V is again an element of V, the product of an element of V by an element of K is an element of V, and the following properties are satisfied: