EPaDel-NJ Fall 2023 Section Meeting – Faculty/Graduate Speaker Session 6, Driscoll 221

Fourier Inversion of Moment Generating Functions (Short Report)

Abstract Moment Generating Functions (MGFs) are insightful in probability theory and statistics, sharing the identical form of a two-sided Laplace transform. In a typical progression of the introduction of MGFs, a look-up table is usually present for the purpose of finding the corresponding probability distribution (PDF), from which an MGF is calculated by summation or integration.

Here, we start from the definitions of integral transform and (exponential) Fourier transform, and delve into the general case of moment generating functions, and then perform a “Fourier inversion” to retrieve the PDF. The Fourier transform is used as a wrapper tool in this process.

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Integral Transform

0. Integral Transform The Lebesgue integral or improper Riemann integral

$$g(y) = \int_{-\infty}^{\infty} K(x, y)\ f(x)\ dx$$

is either real or complex, and is called an integral transform of f, from variable x to variable y; the function K which appears in the integrand is called the kernel of the transform.

1. Exponential Fourier Transform The formula is

$$g(y) = \int_{-\infty}^{\infty} e^{-ixy} f(x)\, dx \tag{1}$$

for which, at points of continuity of f, we have

$$f(x) = \lim_{\alpha \to \infty} \frac{1}{2\pi} \int_{-\alpha}^{\alpha} g(y) e^{ixy}\, dy \tag{2}$$

which is the inverse formula for Fourier transforms, which suggests that a continuous function satisfying the **conditions of the Fourier integral theorem** is uniquely determined by its Fourier transform g.

2. The General Case of Moment Generating Functions For a discrete random variable denoted by X, its moment generating function or "MGF" is a function of a new variable t, by performing the following variable transformation

$$m(t) = E(e^{tX}) = \sum_{\forall y} e^{tx}\, p(x),$$

where the PDF of X integrates/sums to one, $\sum_{\forall x} p(x)\, dx = 1$

General Case of MGF (cont.)

Consider X as a more general, continuous random variable, then its moment generating function is going to be defined by the improper Riemann integral

$$m(t) = \int_{-\infty}^{\infty} e^{tx}\; p(x)\; dx \tag{3}$$

where $p(x)$, the PDF, satisfies the conditions of the Fourier integral theorem and is subject to the constraint of integrating to 1. Then $m(t)$ here is an integral transform of p, from variable x to variable t, in which the kernel of transform is $K(x,t) = e^{tx}$.

3. Fourier Inversion Consider the exponential Fourier transform of some PDF p as follows,

$$g(t) = \int_{-\infty}^{\infty} e^{-itx} p(x)\, dx \tag{4}$$

for which at points of continuity of p we have

$$p(x) = \lim_{\alpha \to \infty} \frac{1}{2\pi} \int_{-\alpha}^{\alpha} g(t) e^{ixt}\, dt \tag{5}$$

so we have MGF's relationship in terms of $g(t)$ to be

$$\begin{aligned} m(t) &= \int_{-\infty}^{\infty} e^{tx} p(x)\, dx \\ &= \int_{-\infty}^{\infty} e^{-i\cdot(it)x} p(x)\, dx \\ &= g(it) \end{aligned}$$

Fourier Inversion (cont.)

and similarly, if we take as input $-it$ in the MGF, it becomes the form of the Fourier transform:

$$m(-it) \;=\; \int_{-\infty}^{\infty} e^{(-it)\cdot x} p(x)\, dx \;=\; g(t)$$

and inverse-transform it, we have

$$p(x) \;=\; \lim_{\alpha\to\infty} \frac{1}{2\pi} \int_{-\alpha}^{\alpha} m(-it)\, e^{ixt}\, dt \tag{6}$$