CHAPTER 1
The How, When, and Why of Mathematics

What is mathematics? Many people think of mathematics (incorrectly) as addition, subtraction, multiplication, and division of numbers. Those with more mathematical training may think of it as dealing with algorithms. But most professional mathematicians think of it as much more than that. While we certainly hope that our students will perform algorithms correctly, what we really want is for them to understand three things: how you do something, why it works, and when it works. The problems we present to you in this book concentrate on these three goals. If this is the first time you have been asked to prove theorems, you may find this to be quite a challenge. Not only will you be learning how to solve the problem, you will also be learning how to write up the solution. The necessary definitions and background to understand a problem, as well as a general plan of attack, will always be presented in the text. It's up to you to spend the time reading, trying various approaches, rereading, and reapproaching. You will probably be spending more time on fewer exercises than you ever have before. While you are now beyond the stage of being given steps to follow and practice, there are general rules that can assist you in your transition to doing higher mathematics. Many people have written about this subject before.

The classic text on how to approach a problem is a wonderful book called *How to Solve It* by George Pólya, [66].

In his text, Pólya gives a list of guidelines for solving mathematical problems. He calls his suggestions "the list." We have included the original in Appendix 28.3. This list has served as a guide for several generations of mathematicians, and we suggest that you let it guide you as well. Here's a closer look at "the list" with some 21st-century modifications.

First. "Understanding the problem." Easier said than done, of course. What should you do? Make sure you know what all the words mean. You may need to look something up in this book, or you may need to use another book. Look at the statement to figure out carefully what you are given and what you are supposed to figure out. If a picture will help, draw it. Will you be proving something? What? Will you have to obtain an example? Of what? Check all conditions. Will you have to show that something is false? Once you understand what you have to do, you can move on to the next step.

Second. "Devising a plan." How will you attack the problem? At this point, you understand what must be done (because you have completed Step 1). Have you seen something like it before? If you haven't looked over class notes, haven't read the text, or haven't done the previous homework assignments, the odds are slim that you have seen anything that will be helpful. Do all that first. Look over the text with the problem in mind, read over your notes with the problem in your head, look at previous exercises and theorems that sound similar. Maybe you can use some of the ideas in the proof of a theorem, or maybe you can use a previous homework problem. Mathematics builds on itself and the problems in the text will also. If you are truly stuck, try to answer a simpler, similar question. Once you decide on a method of approach, try it out.

Third. "Carrying out the plan." Solve the problem. Look at your solution. Is each sentence true? Sometimes it is difficult to catch an error right after you have "found a solution." Put the problem down and come back to it a few hours later. Is each sentence still true?

Fourth. "Looking Back." Pólya suggests checking the result and the argument, or even looking for a different proof. If you are allowed (check with your teacher), one really good way to check a proof is to give it to someone else. You can present it to friends. Even if they don't understand a word you are saying, sometimes saying it out loud in a coherent manner will allow you to recognize an error you can't spot when you are reading. If you are permitted to work together, switch proofs and ask your partner for criticism of your proof.

When you are convinced that your argument is correct, it is time to write up a correct and neat solution to the problem.

Here is an example of the Pólya method at work in mathematics; we will decipher a message. A cipher is a system that is used to hide the meaning of a message by replacing the letters of the alphabet by other letters or symbols.

Exercise 1.1.
The following message is encoded by a shift of the alphabet; that is, every letter is replaced by another one that has been shifted n places further down the alphabet. Once we reach the end of the alphabet, we start over. For instance, if n were 7, we would make the replacements a $\to$ h, b $\to$ i, . . . , s $\to$ z, t $\to$ a, ... Now the exercise: What does the message below say?

PDEO AJYKZEJC WHCKNEPDI EO YWHHAZ W YWAOWN YELDAN. EP EO RANU AWOU PK XNAWG, NECDP?

Let's use the ideas from Pólya's list to solve this. If you have solved problems like this before, it might be a better exercise for you to try on your own to see how this fits Pólya's method before you read on.

1. "*Understanding the problem.*" Each sequence of letters with no blank space between the letters represents one word. Each letter is shifted by the same number of places: namely n. So n is the unknown in this problem and it is what we need to find. Once we know the value of n, we can decipher the whole message. In addition, once we know the meaning of one letter, we can find the value for n.
2. "*Devising a plan.*" A cipher text may have weak points. What are these? How about the short words? Looking at the short words, in some sense, substitutes an easier problem for the one we have.

3. "*Carrying out the plan.*" The short words are:

W;
EO (which appears twice);
EP;
PK.

Try using the most common one and two letter words. For each guess, check the beginning of the cipher text to see if it makes sense. It shouldn't take long for you to come up with the message.

4. "*Looking Back.*" If your solution makes sense, then it is highly unlikely that a different replacement is also possible. So the solution is (with high probability) the only one.

Would there have been other solution methods? Sure. For instance, not all letters have the same frequency in the English language. One analysis of English texts showed the letter e occurring most frequently, followed by (in this order) t, a, o, i, n, s, h, and r. (See [78, p. 19].) We could have used this information to guess the assignment of letters.

We also could have simply tried one value of n after another until the message made sense.

Have you now solved the problem? If you know what the message says, then the answer to this question is *yes*. Are you done? Unless you solved the problem and wrote up a clear, complete solution, the answer to this second question is *no*. A solution consists of a report that tells the reader how you solved the problem and what the answer is. This needs to be done in clear English sentences. As you write up your solution, try to keep the reader in mind. You should explain things clearly and logically, so that the reader doesn't have to spend time filling in gaps. ○

We now move on to a very different kind of example. Consider the set of points in three-space. In case you haven't seen this before, these points are easily described. We take the familiar xy-plane, and place it parallel to the floor. The z-axis is the vertical line perpendicular to the xy-plane and passing through the origin of the xy-plane (see Figure 1.1).

We'll review the important concepts before we begin our example.

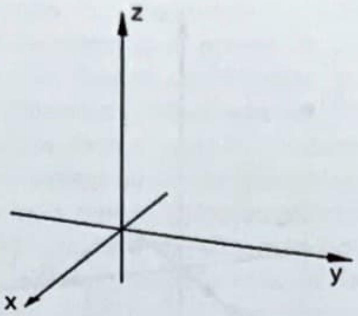

FIGURE 1.1

To locate a point, we will give three coordinates. The first coordinate is the x-coordinate and tells us the number of units to walk in the x-direction. The second is the y-coordinate, telling us how to move in the y-direction and the third is the z-coordinate, telling us how far, up or down, to move. So a point in three-space is denoted by (x, y, z). It is important to make sure you understand this. Try to think of how you would plot points. The point $(1, 0, 0)$ (plotted in Figure 1.2) would appear one unit in the positive direction on the x-axis (since it doesn't move in the y-direction or z-direction at all). The point $(-1, 1, 0)$ would appear in the xy-plane, one unit back on the x-axis and one unit in the positive y-direction. Finally the point $(2, -1, 3)$ is plotted in Figure 1.2.

Let's go a bit further here. In two-space, what was $x = 0$? Since y does not appear in that equation, it is unrestricted and can be any real number. That's why $x = 0$ in two-space is the y-axis. What is $x = 3$? It is a line parallel to the y-axis through the point $(3, 0)$. So, let's try to generalize this to the situation in three-space. What's the plane $z = 0$? Recall that if a variable doesn't appear, then it may assume any value. So this means that z is fixed at 0 while x can take any value, as can y. Thus, the plane $z = 0$ is the xy-plane. Similarly, the yz-plane is the plane $x = 0$ and the xz-plane is the plane $y = 0$. These three planes are called the coordinate planes. What's the plane $z = 3$? $x = 2$? $y = y_0$? There's plenty to think about here, but let's start by asking what the distance is between two points in three-space.

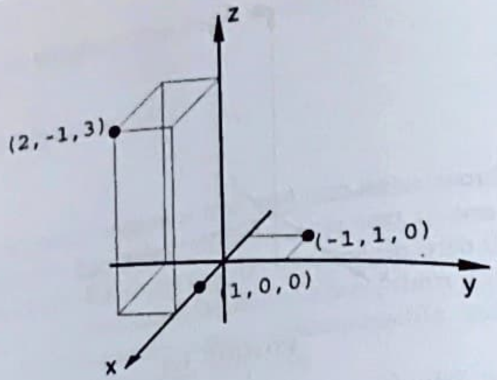

FIGURE 1.2

Example 1.2.
Given two points (x_0, y_0, z_0) and (x_1, y_1, z_1) in three-space, what is the distance between the two points?

We follow Pólya's method to find the solution.

1. "*Understanding the problem.*" Before we begin, we make sure we really understand the meaning of each word and symbol above. We spent the last few paragraphs making sure we all understand the symbols, and all the words are familiar ones that appear in a standard English dictionary. But, wait—has "distance between two points" really been defined? We need to be sure that everyone means the same thing by this. The distance between these two arbitrary points would mean the length of the straight line segment joining the two points. That's what we need to find. What were we given? Two points and their coordinates.
2. "*Devising a plan.*" How do we solve something like this? We haven't covered anything yet, so what can the authors be thinking? If you have no idea how to get started, try thinking about finding the distance between two specific points. Of course, (and this is very important) this won't give us a general formula because it is much too specific, but maybe we'll get some ideas.

So what's the distance between the two points $(1, 0, 0)$ and $(-1, 0, 0)$? That question is easier to answer—it's two. What's

the distance between $(1, 1, 0)$ and $(-1, -2, 0)$? This seems to be just the distance between two points in the familiar xy-plane. We saw a formula for that at some point. It was obtained using the Pythagorean Theorem. What was it? If you can't recall the formula, look it up or (better, yet) try to derive it again.

Our reasoning now brings us to a simpler, similar question. As you recall, this is precisely where Pólya suggested we look for a plan. So far, it seems we can find the distance between two points as long as they lie in a plane parallel to one of the coordinate planes. But in this problem, if we look at the two points, they need not lie in such a plane. We can try to insert a third point that helps us to reduce the problem to one we can already solve. Which point? A picture will help here, so we draw one in Figure 1.3.

We see that (x_0, y_0, z_0) and (x_1, y_1, z_0) lie in the plane $z = z_0$, while (x_1, y_1, z_0) and (x_1, y_1, z_1) lie on the same vertical line, in the intersection of the two planes, $x = x_1$ and $y = y_1$. We "devise our plan" using these three points. Can we get the distance we are looking for from these three points? Look at Figure 1.3 and see if you can guess the rest before going on to Step 3. You probably noticed that the vertical line makes a right angle with every line in the plane $z = z_0$. This should suggest something to you—something like the Pythagorean Theorem.

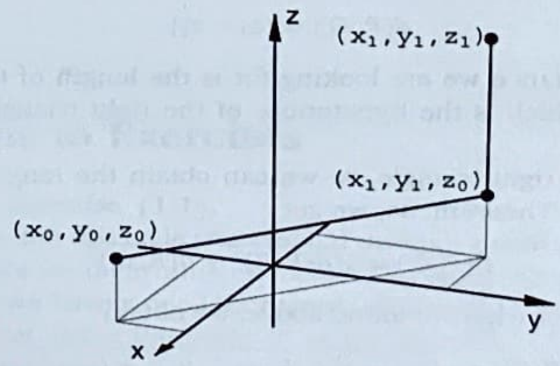

FIGURE 1.3

3. "*Carrying out the plan.*" This is the only thing the reader will see. Everything that preceded this was to assist us in obtaining this solution. That means the reader doesn't know what the points are; we have to tell him or her that. We should make sure we say why a sentence follows from the previous one and we should use equal signs between equal objects. When we think we are done, we should tell the reader that too.

Solution.
Let $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ be two points in space. We claim that the distance between these two points, denoted by $d(P, Q)$, is

$$d(P,Q) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

Proof.
We introduce a third point with coordinates $R = (x_1, y_1, z_0)$. Since (x_0, y_0, z_0) and (x_1, y_1, z_0) both lie in the plane $z = z_0$, we can use the distance formula for two points in a plane to find the distance between them. Thus, the distance is given by

$$d(P,R) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

Now look at the distance between the two points (x_1, y_1, z_0) and (x_1, y_1, z_1). Since these points lie on the same vertical line, the distance is given by

$$d(R,Q) = |z_0 - z_1|.$$

Now, the distance we are looking for is the length of the line segment PQ, which is the hypotenuse of the right triangle PQR (see Figure 1.4).

This is a right triangle, so we can obtain the length using the Pythagorean Theorem. So, we get

$$d(P,Q) = \sqrt{d(P,R)^2 + d(R,Q)^2}.$$

Substituting in what we found above, we obtain

$$d(P,Q) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

This completes the proof. ∎

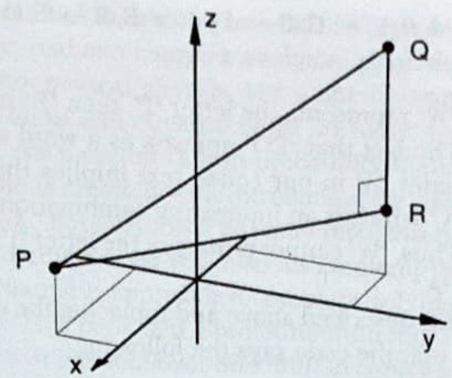

FIGURE 1.4

4. "*Looking back.*" What we have presented is our version of the proof. You may find that you need to include more details. By all means, go ahead. If you had to stop and say, "where did that come from?" make sure you answer yourself. Write it in the text (you aren't going to sell this book back anyway, right?), or keep a notebook of "proofs with commentary." Note that though we used pictures to illustrate the ideas in our argument, a picture will not, in general, substitute for a proof. However, it can really clarify an idea. Don't rely on a picture, but don't be afraid to use one either. ○

Solutions to Exercises

Solution to Exercise (1.1).
We are given that this code was created through a shift of the alphabet. Thus once we determine one letter, the other letters are easily found. Since we have a one-letter word, we'll start with it. Thus "W" must represent either the letter "I" or the letter "A." Checking both shifts of the alphabet

$(W \to I, X \to J, Y \to K, Z \to L, A \to M, B \to N, C \to O,$ etc.

$W \to A, X \to B, Y \to C, Z \to D, A \to E, B \to F, C \to G$, etc.)

we find that if "W" represents the letter "I", then "E" must represent the letter "Q." The fact that "EO" appears as a word and "O" would represent the letter "A" in our coded text implies that "EO" would be the word "QA," which is an interesting combination of letters, but hardly a word. Thus, "W" cannot represent the letter "I" and therefore "W" represents "A."

Using the shift described above and replacing the corresponding letters, we find that the code says the following.

"THIS ENCODING ALGORITHM IS CALLED A CAESAR CIPHER. IT IS VERY EASY TO BREAK, RIGHT?"

In fact, the Caesar cipher is quite easy to break. If this interests you, a very readable history of coding theory is presented by S. Singh in *The Code Book*, [78].

Spotlight: George Pólya

György Pólya (1887–1985), referred to as George Pólya in his later years, was born and raised in Hungary. He studied in Vienna and in Budapest, where he received his doctorate in 1912. One of his influential teachers was Leopold Fejér. In his book [67, p. 39], Pólya refers to Fejér as "an inspiring teacher who had a great deal of influence on Hungarian mathematicians of the time." The two primary places that Pólya taught were the Eidgenössische Technische Hochschule (ETH) in Zürich, Switzerland and Stanford University in Palo Alto, California.

Though Pólya's mother tongue was Hungarian, he worked in the Swiss-German speaking part of Switzerland and he spoke French with his wife from Neuchâtel, a city in the French speaking part of Switzerland. In school he also learned Latin and Greek. (See [67, p. 11].) Pólya later emigrated to the United States where he taught and lectured in English. He published mathematical papers in Hungarian, German, French, English, Italian, and Danish.

Pólya contributed to original research in probability, geometry, number theory, real and complex analysis, graph theory, combinatorics, and mathematical physics. His name is connected to many mathematical ideas and constructions. To name just a few of his achievements, we mention that in probability there is a Pólya distribution and he is credited with introducing the idea and the term of "random walk." But Pólya was not only recognized as an excellent scholar of mathematics, he was also an excellent teacher of mathematics. His heuristic approach to problem solving is outlined in *How to Solve It*. This book had a profound influence on the teaching of mathematics. It has sold over one million copies and is translated into over 20 languages. Records kept by the ETH in Zürich show that Pólya was the advisor of 14 thesis students there and, according to [62], he was the advisor of 9 more students at Stanford.

The Mathematical Association of America (MAA) gives an annual Pólya award. According to the MAA website, "This award, established in 1976, is named after the renowned teacher and writer, and is given for articles of expository excellence published in the *College Mathematics Journal*."

To learn more about George Pólya and his approach to problem solving, we recommend reading his book *How to Solve It*, [66], the picture book [67] (which contains a short biography), or consulting the more in-depth account of Pólya's life, written by his former student at Stanford, [3]. The article [85] is based on interviews with Pólya and appeared in an issue of *Mathematics Magazine* entirely devoted to Pólya and his work.

Problems

Problem 1.1.
Here is a problem intended to help you work through "the list." After this, you are on your own.

Find a word (written in standard capital letters) that is unchanged when reflected in a horizontal line and in a vertical line. The word must appear in a dictionary (in a language of your choice) in order to be a valid solution.

1. "*Understanding the problem.*" We need to find a word. We are given information about the letters that make up this word. There are two conditions: Two different reflections should not alter the word.
   Try these two reflections on a word, say on SOLUTION, to make sure you understand the problem.
2. "*Devising a plan.*" We have to find the connection between what we are given and what we have to find.
   Which letters of the alphabet satisfy each of the two conditions? Both conditions?
   Find a word that is not changed if it is reflected in a horizontal line.
   Find a word that is not changed if it is reflected in a vertical line.
   Formulate the exact conditions for this exercise; that is, state the letters that can be used and how they must be arranged.
3. "*Carrying out the plan.*" Find a word that satisfies the conditions given above.
4. "*Looking Back.*" Are there other solutions?

Problem 1.2.
Find a word (written in standard capital letters) that reads the same forward and backward and is still the same forward and backward when rotated around its center 180°. Your solution needs to appear in a standard dictionary of some language.

Problem 1.3.
Solve the following anagrams. The first three are places (in the geographical sense), and the fourth is a place you might live in. All can be rearranged to form a single word.

(a) NOVA CURVE;
(b) NINE SLAP NAVY;
(c) I HELD A HIP PAL;
(d) DIRTY ROOM.

Note: You may have to find out exactly what an anagram is. This is part of Pólya's first point on the list.

Problem 1.4.
Suppose n teams play in a single game elimination tournament. How many games are played?

An example of such tournaments are the various categories of the U. S. Open tennis tournament; for example, women's singles.

Note: Pay special attention to the first entry of Pólya's list: "Is it possible to satisfy the condition?"

Problem 1.5.
Suppose you are all alone in a strange house. There are seven identical closed doors. The bathroom is behind exactly one of them. Is it more likely, less likely, or equally likely that you find the bathroom on the first try than on the third try? Why?

Problem 1.6.
The following message is encoded using a shifted alphabet just as in Exercise 1.1. (Of course, the shift number n is not the same as in the exercise!) What does the message say?

RDSXCVIWTDGNXHUJCLTLXAAATPGCBDGTPQDJIXIAPITG

Problem 1.7.
Give a detailed description of all points in three-space that are equidistant from the x-axis and the yz-plane. Once you decide on the answer, write the solution up carefully. Pay particular attention to your notation.

Problem 1.8.
The following is a classic problem in mathematics. Though there are many variations of this problem, the standard one is the following.

You are given 12 coins that appear to be identical. However, one of the coins is counterfeit, and the weight of this coin is slightly different than that of the other 11. Using only a two-pan balance, what is the smallest number of weighings you would need to find the counterfeit coin? (Think about a simpler, similar problem.)

(See I. Peterson's web site [64] for a discussion of this problem.)

Problem 1.9.
Let n be an odd integer. Prove that $n^3 - n$ is divisible by 24.

The following two problems are only appropriate if you took at least two semesters of calculus. Though you may have worked these before, the idea is to work them again paying close attention to the final presentation. Make sure you define all variables. Use complete sentences, with proper punctuation.

Problem 1.10.
Find the volume of a spherical cap if the height is 2 m and the radius of the rim of the cap is 5 m.

Problem 1.11.
We have two circular right cylinders of radius 1 each. The axes of the two cylinders intersect at a right angle. Find the volume of the solid that both cylinders have in common.

Tips on Doing Homework

Your instructor will probably ask you to work many of the exercises and problems in this text. If there is one thing mathematicians agree on, it is that you learn mathematics by doing it. Here are some tips on how to get started.

- Make sure you know what the rules are. Some instructors do not want you to get help from someone else. Other instructors encourage working together in groups. Ask, if you are not clear about the policy.
- If you are permitted to work together, form a study group. A small group of two to four people usually works best. Get together on a regular basis and discuss the assigned problems.
- Read the questions carefully. If there is a term that you do not know, look it up.
- Before you get started, read over the text and the notes from class, paying particular attention to definitions, theorems, and previous exercises. It isn't unusual to spend several hours on a single problem at this point. Doing mathematics means pondering a problem for hours, days, weeks, even years (though we have tried not to

pose problems that will take you years to solve). Working two hours on one problem, thinking about it as you go through your day and then spending another two hours on it the next day is fairly common practice for students at this level.
- Once you have read over the text, looked over the relevant definitions, worked through the examples and tried to solve the problem, you will be well on your way towards understanding the problem. If you can't get started, at least you will know which questions to ask. Seek help from your instructor or other students (if your instructor allows this).
- Once you have a solution to a problem, look at it critically. Check that it is correct. Put it down. Come back to it later. Do you still understand everything? Is it still correct? (As you can imagine, this is very important.) Can you simplify it? If you work with someone else have them read it over. *Never hand in your first draft of a solution to a problem.*
- Writing a solution means convincing a reader that the result is correct. There can be *no* gaps or errors. Explain each step—don't assume that the reader knows what you are thinking. Keep a reader in mind as you write, and remember that the instructor or anyone else who already knows the solution is *not* really your target audience. Though that may be the person for whom the solution is intended, it is your job to convince the reader that each step in your solution is correct. Perhaps a better audience to keep in mind is someone who knows the material from the class, but not the solution to the problem.
- Write up your final solution very carefully and neatly. The reader shouldn't find him or herself proving things for you—you should do that for him or her. Staple pages together so that the reader may have the pleasure of reading your entire proof in the correct order and its entirety.