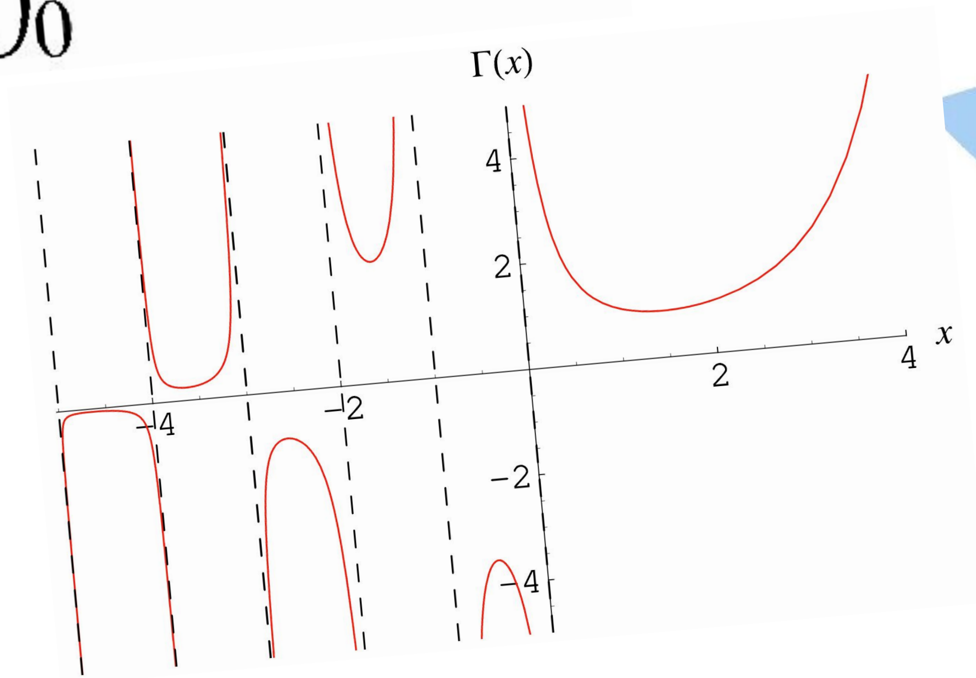
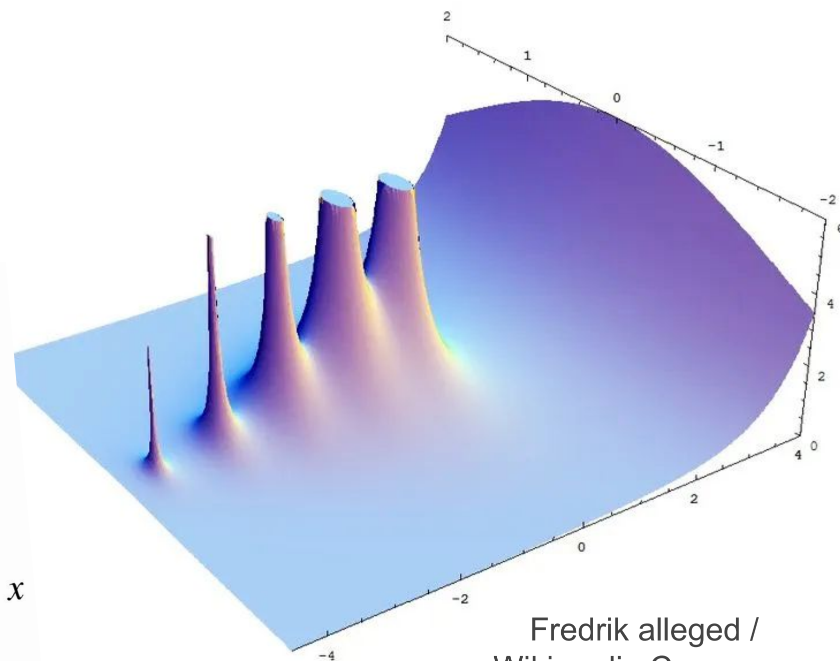
Friday Lunch with the Mathematics Student Association

Properties of the Gamma Function

$$\int_0^\infty t^{x-1} e^{-t}\, dt$$





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"Improper Integrals"

There are two cases in which the meaning of a definite integral is not immediately clear:

- limit of integration is infinite
- domain over which the integral is evaluated contains points where the **function's value** becomes infinite at a vertical asymptote

These situations are improper integrals, and they are evaluated using **limits.**

The Gamma function is defined for all $x > 0$ as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \tag{1}$$

Part (a). Convergence

Want to show that Equation 1 makes sense, namely showing that this improper integral converges.

A hint here is to first prove that

$$\lim_{t\to\infty} t^p e^{-\frac{1}{2}t} = 0$$

for all real numbers p, and then use this to show that there exists a number $M > 0$, which depends on p, such that $0 \leqslant t^p e^{-\frac{1}{2}t} \leqslant M$ whenever $t \geqslant 1$, and then conclude that

$$\int_1^\infty t^{x-1} e^{-t}\, dt$$

exists for all $x > 0$.

Proof **Step #1: Showing that Limit of $t^p e^{-\frac{1}{2}t}$ as $t \to \infty$ Goes to Zero.**

If $p < 0$, then $\lim_{t\to\infty} t^p e^{-\frac{1}{2}t} = 0 \cdot 0 = 0$;

If $p = 0$, then $\lim_{t\to\infty} t^p e^{-\frac{1}{2}t} = \lim_{t\to\infty} t^0 \cdot e^{-\frac{1}{2}t} = 0$;

If $p > 0$, then

$$\lim_{t\to\infty} t^p e^{-\frac{1}{2}t} = \lim_{t\to\infty} \frac{t^p}{e^{\frac{1}{2}t}}$$

by applying l'Hopital's Rules $\lceil p \rceil$ times, the limit of the numerator becomes

$$\lim_{t\to\infty} p(p-1)\cdot\dots\cdot t^{p-\lceil p \rceil} = 0$$

so the entire limit goes to 0, too.

Step #2: Showing $t^p e^{-\frac{1}{2}t}$ Is Upper-Bounded

To begin, $\lim_{t\to\infty} t^p e^{-\frac{1}{2}t} = 0$ implies that when t gets large enough, $t^p e^{-\frac{1}{2}t}$ gets very small.

Then, given any $\epsilon > 0$, $\exists x_0 > 0$ such that for $t > x_0$, we have

$$t^p e^{-\frac{1}{2}t} \; < \; \epsilon$$

is bounded by ϵ; for the **closed interval** $1 \leqslant t \leqslant x_0$, the continuous function $t^p e^{-\frac{1}{2}t}$ is bounded, and let's say it is bounded by M_1.

Overall, when $t \geqslant 1 > 0$, take $M = \max\{M_1, \epsilon\}$ be the bound, i.e.

$$0 \; \leqslant \; t^p e^{-\frac{1}{2}t} \; \leqslant \; M$$

Step #3: Showing Two "Portions" of $\Gamma(x)$ Are Both Upper-Bounded

Continuing on, the corresponding $(1, \infty)$ "portion" of the Gamma function integral can be written and shown bounded:

$$\begin{aligned} \int_1^{\infty} \underline{t^{x-1}\, e^{-\frac{1}{2}t}} \cdot e^{-\frac{1}{2}t}\, dt &\leqslant \int_0^{\infty} M \cdot e^{-\frac{1}{2}t}\, dt \\ &= \lim_{r\to\infty} M(-2) \cdot e^{-\frac{1}{2}t}\Big|_1^r \\ &= -2M \lim_{r\to\infty} \left(e^{-\frac{1}{2}r} - e^{-1/2}\right) \\ &= -2M(0 - e^{-1/2}) \\ &= 2e^{-1/2}M; \end{aligned}$$

In parallel, the remaining $(0,1]$ portion is **convergent**,

$$\int_0^1 t^{x-1}e^{-t}\,dt \leqslant \int_0^1 t^{x-1}\cdot(1)\,dt$$

which is convergent for $x-1>-1$ (since we're given $x>0$), so it's upper-bounded by the value it converges to.

Overall, because both portions of $\Gamma(x)$ are upper-bounded, $\Gamma(x)$ is upper-bounded and convergent. $\square$

Part (b). Show that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$

$$
\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^{(x+1)-1} e^{-t}\, dt \\
&= \int_0^\infty t^x e^{-t}\, dt \\
\text{integration by parts ....} &= \int_0^\infty t^x (-1)\, d(e^{-t}) \\
&= -\left(t^x e^{-t} \Big|_0^\infty - \int_0^\infty e^{-t}\, dt^x \right) \\
&= \int_0^\infty e^{-t} x t^{x-1}\, dt - \lim_{r\to\infty} r^x e^{-r} \Big|_0^r \\
&= x \int_0^\infty e^{-t} t^{x-1}\, dt - \lim_{r\to\infty} (r^x e^{-r} - 0) \\
\text{l'Hopital's rules ....} &= x\Gamma(x) - \lim_{r\to\infty} \frac{r^x}{e^r} \\
&= x\Gamma(x) - 0 \\
&= x\Gamma(x). \qquad \square
\end{aligned}
$$

Part (c). Show that $\Gamma(n+1) = n!$ for each natural number n

To begin,

$$\begin{aligned}\Gamma(1) &= \int_0^\infty t^{1-1}e^{-t}\,dt \\ &= \int_0^\infty e^{-t}\,dt \\ &= -\lim_{r\to\infty} e^{-t}\Big|_0^r \\ &= -\lim_{r\to\infty}(e^{-r}-1) \\ &= -(0-1) = 1;\end{aligned}$$

and by result of part (b),

$$\Gamma(2) = 1\cdot\Gamma(1) = 1\cdot 1 = 1$$

so as a base case for $n=1$, $\Gamma(1+1) = 1! = 1$ holds.

and the induction completes for $n = k$ (hypothesis) and $n = k + 1$

Part (d). Show that $\int_0^\infty e^{-u^2}\ du = \frac{1}{2}\sqrt{\pi}$ using double integrals in polar coordinates

Because e^{-u^2} is an even function, we write

$$
\begin{aligned}
\int_0^\infty e^{-u^2}\ du &= \frac{1}{2}\int_{-\infty}^\infty e^{-u^2}\ du \\
&= \frac{1}{2}\sqrt{\left(\int_{-\infty}^\infty e^{-u^2}\ du\right)^2} \quad \cdots\cdots \text{ integral is positive} \\
&= \frac{1}{2}\sqrt{\int_{-\infty}^\infty e^{-x^2}\ dx \int_{-\infty}^\infty e^{-y^2}\ dx} \\
&= \frac{1}{2}\sqrt{\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2}\ dxdy} \\
&= \frac{1}{2}\sqrt{\int_0^{2\pi} \int_0^\infty e^{-r^2}\ r\ drd\theta} \quad \cdots\cdots \text{ using polar coordinates} \\
&= \frac{1}{2}\sqrt{\int_0^{2\pi} d\theta \cdot \int_0^\infty e^{-r^2}\ r\ dr} \\
&= \frac{1}{2}\sqrt{2\pi \cdot (\frac{1}{2})} \\
&= \frac{1}{2}\sqrt{\pi}
\end{aligned}
$$

Part (e). Calculate $\Gamma(1/2)$ and $\Gamma(3/2)$

$$\Gamma(\frac{1}{2}) = \int_0^\infty t^{1/2-1} e^{-t}\, dt$$

let $t = u^2$, then $dt = 2u\ du$

$$= \int_0^\infty (u^2)^{-1/2} e^{-u^2} \cdot 2u\ du$$

$$= 2 \int_0^\infty e^{-u^2}\ du$$

$$= 2 \cdot \frac{1}{2}\sqrt{\pi}$$

$$= \sqrt{\pi};$$

So then

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\,\Gamma(\frac{1}{2}) = \frac{1}{2}\,\sqrt{\pi}.$$