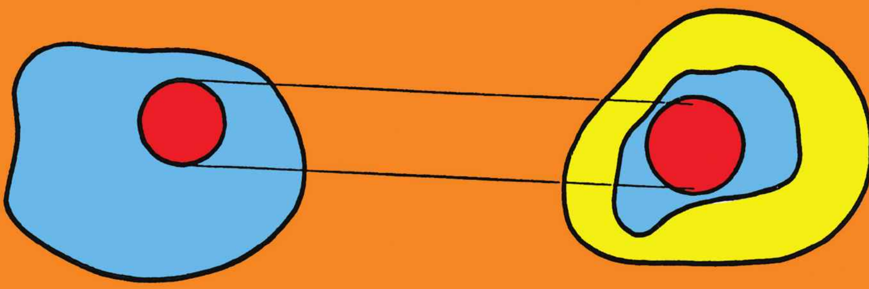
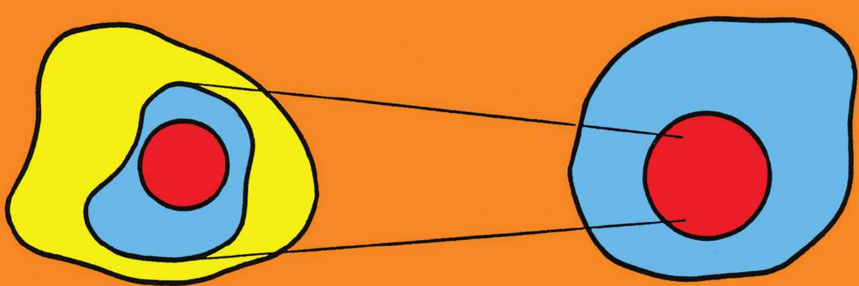


Bert Mendelson

INTRODUCTION TO TOPOLOGY

Third Edition



Introduction to Topology

Third Edition

by Bert Mendelson

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Contents

CHAPTER 2

Metric Spaces

1 INTRODUCTION

A metric space is a set of points and a prescribed quantitative measure of the degree of closeness of pairs of points in this space. The real number system and the coördinate plane of analytic geometry are familiar examples of metric spaces. Starting from the vague characterization of a continuous function as one that transforms nearby points into points that are themselves nearby, we can, in a metric space, formulate a precise definition of continuity. Although this definition may be stated in the so-called "ε, δ" terminology, there are other, equivalent formulations available in a metric space. These include characterizations of continuity in terms of the behavior of a function with respect to certain subsets called neighborhoods of a point, or with respect to certain subsets called open sets.

2 METRIC SPACES

Given two real numbers a and b, there is determined a non-negative real number, $|a - b|$, called the distance between a and b. Since to each ordered pair (a, b) of real numbers there is associated the real number $|a - b|$, we may write this correspondence in functional notation by setting

$$d(a, b) = |a - b|.$$

Thus we have a function $d: R \times R \to R$, where R is the set of real numbers. This function has four important properties, which the reader should verify:

1. $d(x, y) \geqq 0$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) = d(y, x)$;
4. $d(x, z) \leqq d(x, y) + d(y, z)$;

for $x, y, z \in R$. For the purposes of discussing "continuity" of functions, these four properties of "distance" are sufficient. This fact suggests the possibility of examining "continuity" in a more general setting; namely, in terms of any set of points for which there is defined a "distance function" such as the function $d: R \times R \to R$ above.

DEFINITION 2.1 A pair of objects (X, d) consisting of a non-empty set X and a function $d: X \times X \to R$, where R is the set of real numbers, is called a *metric space* provided that:

1. $d(x, y) \geqq 0, \quad x, y \in X$;
2. $d(x, y) = 0$ if and only if $x = y, \quad x, y \in X$;
3. $d(x, y) = d(y, x), \quad x, y \in X$;
4. $d(x, z) \leqq d(x, y) + d(y, z), \quad x, y, z \in X$.

The function d is called a *distance function* or *metric* on X and the set X is called the *underlying set.*

[A more precise notation for a metric space would be $(X, d{:}X \times X \to R)$ and for a distance function $d{:}X \times X \to R$. We shall, however, frequently delete the sets and arrow in the symbol for a function, when, in a given context, it is clear which sets are involved.]

We may think of the distance function d as providing a quantitative measure of the degree of closeness of two points. In particular, the inequality $d(x, z) \leqq d(x, y) + d(y, z)$ may be thought of as asserting the transitivity of closeness; that is, if x is close to y and y is close to z, then x is close to z.

Let $a, b \in R$, where R is the set of real numbers. The verification that the function $d(a, b) = |a - b|$ satisfies the four properties enumerated in Definition 2.1 establishes:

THEOREM 2.2 (R, d) is a metric space, where d is the function defined by the correspondence $d(a, b) = |a - b|$, for $a, b \in R$.

Given a finite collection $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ of metric spaces, there is a standard procedure for converting the set

$$X = \prod_{i=1}^{n} X_i$$

into a metric space; that is, for defining a distance function on X.

THEOREM 2.3 Let metric spaces $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ be given and set

$$X = \prod_{i=1}^{n} X_i.$$

For each pair of points $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in X$, let $d{:}X \times X \to R$ be the function defined by the correspondence

$$d(x, y) = \underset{1 \leqq i \leqq n}{\text{maximum}} \{d_i(x_i, y_i)\}.$$

Then (X, d) is a metric space.

Proof. With x and y as above, $d_i(x_i, y_i) \geqq 0$ for $1 \leqq i \leqq n$, and therefore $d(x, y) \geqq 0$. If $d(x, y) = 0$, then $d_i(x_i, y_i) = 0$ for $1 \leqq i \leqq n$ and therefore $x_i = y_i$ for each i. Consequently, $x = y$. Conversely, if $x = y$, then $d_i(x_i, y_i) = 0$ for each i, and $d(x, y) = 0$. Since $d_i(x_i, y_i) = d_i(y_i, x_i)$ for $1 \leqq i \leqq n$, $d(x, y) = d(y, x)$. Finally, let $z = (z_1, z_2, \ldots, z_n) \in X$. Let j and k be integers such that $d(x, y) = d_j(x_j, y_j)$ and $d(y, z) = d_k(y_k, z_k)$. Thus, for $1 \leqq i \leqq n$, $d_i(x_i, y_i) \leqq d_j(x_j, y_j)$, $d_i(y_i, z_i) \leqq d_k(y_k, z_k)$, and

$$d_i(x_i, z_i) \leqq d_i(x_i, y_i) + d_i(y_i, z_i) \leqq d_j(x_j, y_j) + d_k(y_k, z_k) = d(x, y) + d(y, z).$$

Therefore $d(x, z) = \underset{1 \leqq i \leqq n}{\text{maximum}} \{d_i(x_i, z_i)\} \leqq d(x, y) + d(y, z)$.

As an immediate application of this theorem, we have:

COROLLARY 2.4 (R^n, d) is a metric space, where $d: R^n \times R^n \to R$ is the function defined by the correspondence

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \underset{1 \leqq i \leqq n}{\text{maximum}} \{|x_i - y_i|\}, \quad (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in R^n.$$

It is interesting to compare the metric space (R^2, d) that we obtain in the above manner with what might be considered a more natural model of the coordinate plane. In (R^2, d) as defined above, the distance from the point $(1, 2)$ to the point $(3, 1)$ is 2, since maximum $\{|1 - 3|, |2 - 1|\} = 2$. The distance function d' used in analytical geometry would yield

$$d'((1, 2), (3, 1)) = \sqrt{(1 - 3)^2 + (2 - 1)^2} = \sqrt{5}.$$

If, for each pair of points $(x_1, x_2), (y_1, y_2) \in R^2$ we define

$$d'((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

then we are constructing a new metric space (R^2, d'), (provided, of course, that d' is a distance function), which must be distinguished from the metric space (R^2, d) where

$$d((x_1, x_2), (y_1, y_2)) = \text{maximum } \{|x_1 - y_1|, |x_2 - y_2|\}.$$

For example, in (R^2, d) the set M of points x such that $d(x, a) \leqq 1$ for a fixed point $a \in R^2$ is a square of width 2 whose center is at a and whose sides are parallel to the coordinate axes, whereas in (R^2, d') the set of points x such that $d'(x, a) \leqq 1$ for a fixed point $a \in R^2$ is a circular disc whose center is a and whose radius is 1 (see Figure 4).

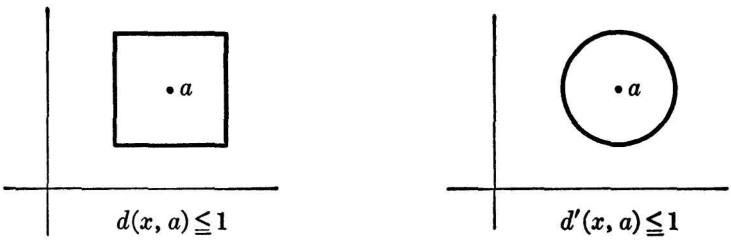


Figure 4

The formula used to define the function d' may be generalized to yield a distance function for R^n, often referred to as the *Euclidean* distance function.

THEOREM 2.5 (R^n, d') is a metric space, where d' is the function defined by the correspondence

$$d'(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in R^n$.

The proof of this theorem will be found in Section 8.

The fact that we have metric spaces (R^n, d) and (R^n, d'), with d and d' defined as above, serves to emphasize the fact that a metric space consists of two objects, a set and a distance function. Two metric spaces may be distinct even though the underlying sets of points of the two spaces are the same.

EXERCISES

1. Let (X, d) be a metric space. Let k be a positive real number and set $d_k(x, y) = k \cdot d(x, y)$. Prove that (X, d_k) is a metric space.
2. Prove that (R^n, d'') is a metric space, where the function d'' is defined by the correspondence

$$d''(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in R^n$. In (R^2, d'') determine the shape and position of the set of points x such that $d''(x, a) \leqq 1$ for a point $a \in R^2$.
3. Let d be the distance function defined on R^n by using Theorem 2.3, let d' be the Euclidean distance function, and let d'' be the distance function defined in Problem 2 above. Prove that for each pair of points $x, y \in R^n$,

$$d(x, y) \leqq d'(x, y) \leqq \sqrt{n}\, d(x, y),$$
$$d(x, y) \leqq d''(x, y) \leqq n \cdot d(x, y).$$

4. Let X be the set of all continuous functions $f:[a, b] \to R$. For $f, g \in X$, define

$$d(f, g) = \int_a^b |f(t) - g(t)|\, dt.$$

Using appropriate theorems from Calculus, prove that (X, d) is a metric space.
5. Let $S \subset R$. A function $f:S \to R$ is called *bounded* if there is a real number K such that $|f(x)| \leqq K$, $x \in S$ (or equivalently, $f(S) \subset [-K, K]$). Let X' be the set of all bounded functions $f:[a, b] \to R$. For $f, g \in X'$ define

$$d'(f, g) = \text{l.u.b.} \bigcup_{x \in [a,b]} \{|f(x) - g(x)|\},$$

(l.u.b. is an abbreviation of *least upper bound*, see Definition 5.5 of this chapter). Prove that (X', d') is a metric space.
6. Let $f, g:[a, b] \to R$ be two functions that are both continuous and

bounded. Compare $d(f, g)$ and $d'(f, g)$, where d and d' are defined as in Problems 4 and 5 respectively.

7. Let X be a set. For $x, y \in X$ define the function d by

$$d(x, x) = 0,$$

and

$$d(x, y) = 1,$$

if $x \neq y$. Prove that (X, d) is a metric space.

8. Let Z be the set of integers. Let p be a positive prime integer. Given distinct integers m, n there is a unique integer $t = t(m, n)$ such that $m - n = p^t \cdot k$, where k is an integer not divisible by p. Define a function $d: Z \times Z \to R$ by the correspondence $d(m, m) = 0$ and

$$d(m, n) = \frac{1}{p^t}$$

for $m \neq n$. Prove that (Z, d) is a metric space. [*Hint:* for $a, b, c \in Z$, $t(a, c) \geqq \text{minimum } \{t(a, b), t(b, c)\}$]. Let $p = 3$. What is the set of elements $x \in Z$ such that $d(0, x) < 1$? What is the set of elements $x \in Z$ such that $d(0, x) < \frac{1}{3}$?

3 CONTINUITY

In calculus, the first occurrence of the word "continuity" is with reference to a function $f: R \to R$, R the set of real numbers. To decide which condition or conditions this function must satisfy for us to say, "the function f is continuous at a point $a \in R$," we try to decide upon a precise formulation of the statement "a number $f(x)$ will be close to the number $f(a)$ whenever the number x is close to a." Having defined a distance function for the real numbers R, we have a quantitative measure of the degree of closeness of two numbers. But how close must $f(x)$ be to $f(a)$? Instead of specifying some particular degree of closeness of $f(x)$ to $f(a)$, let us think, rather, of requiring that no matter what choice is made for the degree of closeness of $f(x)$ to $f(a)$, it can be so arranged that this degree of closeness is achieved. By the phrase "arrange matters" we mean that we can find a corresponding degree of closeness so that whenever x is within this corresponding degree of closeness to a, then $f(x)$ is within the prescribed degree of closeness to $f(a)$. We have now arrived at the following formulation, "the function $f:R \to R$ is continuous at the number $a \in R$, if given a prescribed degree of closeness, $f(x)$ will be within this prescribed degree of closeness to $f(a)$, whenever x is within some corresponding degree of closeness to a." To put this statement in its final form, we shall substitute for "a prescribed degree of closeness" the symbol "ε," and for the phrase "some corresponding degree of closeness" the symbol "δ," and use the distance function to measure the degree of closeness.

Definition 3.1 Let $f:R \to R$. The function f is said to be *continuous at the point* $a \in R$, if given $\varepsilon > 0$, there is a $\delta > 0$, such that

$$|f(x) - f(a)| < \varepsilon,$$

whenever

$$|x - a| < \delta.$$

The function f is said to be *continuous* if it is continuous at each point of R.

Because we initially formulated the definition of continuity in terms of the phrase "degree of closeness," we may easily devise a definition of "continuity" applicable to metric spaces in general, since we need only use the distance functions of these metric spaces to measure "degree of closeness."

Definition 3.2 Let (X, d) and (Y, d') be metric spaces, and let $a \in X$. A function $f:X \to Y$ is said to be *continuous at the point* $a \in X$ if given $\varepsilon > 0$, there is a $\delta > 0$, such that

$$d'(f(x), f(a)) < \varepsilon$$

whenever $x \in X$ and

$$d(x, a) < \delta.$$

The function $f:X \to Y$ is said to be *continuous* if it is continuous at each point of X.

Definitions, such as those given above, are created to serve two purposes. First of all, they are abbreviations. Thus, the statement that begins, "given $\varepsilon > 0$, there is . . . ," is replaced by the shorter statement, "$f:X \to Y$ is continuous at the point $a \in X$." Second, these definitions are attempts to formulate precise characterizations of what we feel are significant properties; in this case, the property of being continuous at a point. We have tried to indicate in the discussion preceding these definitions that they do provide a precise characterization of our intuitive, and perhaps vague, concept of continuity. There are, in a certain sense, tests that we may apply to see whether or not they do so. As an illustration, there are certain functions that we "know" are "continuous," that is, we are sure that they possess this property we are trying to characterize. If it should turn out that a function we "know" to be "continuous" is not continuous in accordance with these definitions, then, although these definitions may be precise, they would not furnish a precise characterization of the property we have in mind when we say a function is "continuous." This type of testing of a definition thus takes the form of proving theorems to the effect that certain functions are continuous. For example:

Theorem 3.3 Let (X, d) and (Y, d') be metric spaces. Let $f:X \to Y$ be a constant function, then f is continuous.

Proof. Let a point $a \in X$ and $\varepsilon > 0$ be given. Choose any $\delta > 0$, say $\delta = 1$. Then whenever $d(x, a) < \delta$, we have $d'(f(x), f(a)) = 0 < \varepsilon$.

Theorem 3.4 Let (X, d) be a metric space. Then the identity function $i:X \to X$ is continuous.

Proof. Suppose $a \in X$. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$, then whenever $d(x, a) < \delta$ we have $d(i(x), i(a)) = d(x, a) < \varepsilon$.

Note that in the above proof we could have equally well chosen δ to be any positive number, provided only that $\delta \leqq \varepsilon$, and the proof would still be valid. The choice of δ need not be a very efficient choice; all that is required is that it "do the job."

There is one situation we shall have to consider for which the notation $f: X \to Y$ that we have adopted for a function from a metric space (X, d) into a metric space (Y, d') is ambiguous. Consider metric spaces (X, d) and (X, d') with the same underlying set. If we simply write $f: X \to X$ for a function, it is impossible to tell which metric space is denoted by the first occurrence of X and which by the second. For this reason, when considering one set X with two different distance functions, we shall write $f: (X, d) \to (X, d')$ if we intend to think of $f: X \to X$ as a function from the metric space (X, d) into the metric space (X, d'). As an illustration, we shall prove:

THEOREM 3.5 Let $i: R^n \to R^n$ be the identity function. Then

$$i: (R^n, d) \to (R^n, d')$$

and

$$i: (R^n, d') \to (R^n, d)$$

are continuous, where the distance function d is the maximum distance between corresponding coordinates (as defined in Section 2) and d' is the Euclidean distance.

Proof. Let $a = (a_1, a_2, \ldots, a_n) \in R^n$. We shall first prove that $i: (R^n, d) \to (R^n, d')$ is continuous. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/\sqrt{n}$. Suppose $x = (x_1, x_2, \ldots, x_n)$ is such that $d(x, a) < \delta$; that is, $\underset{1 \leqq i \leqq n}{\text{maximum}} \{|a_i - x_i|\} < \delta$. Then

$$d'(x, a) = \sqrt{\sum_{i=1}^{n} (a_i - x_i)^2} < \sqrt{n\delta^2} = \sqrt{\varepsilon^2} = \varepsilon.$$

Therefore, given $\varepsilon > 0$, there is a $\delta > 0$ such that $d'(i(x), i(a)) < \varepsilon$ whenever $d(x, a) < \delta$.

We now prove that $i: (R^n, d') \to (R^n, d)$ is continuous. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Suppose that $x = (x_1, x_2, \ldots, x_n)$ is such that $d'(x, a) < \delta$. Then

$$\sum_{i=1}^{n} (a_i - x_i)^2 < \delta^2$$

and therefore for each i, $(a_i - x_i)^2 < \delta^2$, or $|a_i - x_i| < \delta = \varepsilon$. Consequently, $d(x, a) < \varepsilon$. Thus, given $\varepsilon > 0$, there is a $\delta > 0$, such that $d(i(x), i(a)) < \varepsilon$ whenever $d'(x, a) < \delta$.

One of the most important elementary theorems about continuous functions is the statement that the composition of two continuous functions is again a continuous function.

THEOREM 3.6 Let (X, d), (Y, d'), (Z, d'') be metric spaces. Let $f:X \to Y$ be continuous at the point $a \in X$ and let $g:Y \to Z$ be continuous at the point $f(a) \in Y$. Then $gf:X \to Z$ is continuous at the point $a \in X$.

Proof. Let $\varepsilon > 0$ be given. We must find a $\delta > 0$ such that whenever $x \in X$ and $d(x, a) < \delta$, then $d''(g(f(x)), g(f(a))) < \varepsilon$. Since g is continuous at $f(a)$, there is an $\eta > 0$, such that whenever $y \in Y$ and $d'(y, f(a)) < \eta$, then $d''(g(y), g(f(a))) < \varepsilon$. Using the fact that f is continuous at a, we know that given $\eta > 0$, there is a $\delta > 0$, such that $x \in X$ and $d(x, a) < \delta$ imply that $d'(f(x), f(a)) < \eta$ and hence $d''(g(f(x)), g(f(a))) < \varepsilon$.

COROLLARY 3.7 Let (X, d), (Y, d'), (Z, d'') be metric spaces. Let $f:X \to Y$ and $g:Y \to Z$ be continuous. Then $gf:X \to Z$ is continuous.

EXERCISES

1. Let X be the set of continuous functions $f:[a, b] \to R$. Let d^* be the distance function on X defined by

$$d^*(f, g) = \int_a^b |f(t) - g(t)|\, dt,$$

for $f, g \in X$. For each $f \in X$, set

$$I(f) = \int_a^b f(t)\, dt.$$

Prove that the function $I:(X, d^*) \to (R, d)$ is continuous.

2. Let (X_i, d_i), (Y_i, d_i'), $i = 1, \ldots, n$ be metric spaces. Let $f_i: X_i \to Y_i$, $i = 1, \ldots, n$ be continuous functions. Let

$$X = \prod_{i=1}^{n} X_i \quad \text{and} \quad Y = \prod_{i=1}^{n} Y_i$$

and convert X and Y into metric spaces in the standard manner. Define the function $F: X \to Y$ by

$$F(x_1, x_2, \ldots, x_n) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n)).$$

Prove that F is continuous.

3. Define the function $f: R^2 \to R$ by $f(x_1, x_2) = x_1 + x_2$. Prove that f is continuous, where the distance function on R^2 is either d or d'.

4. Define functions g, h, k, m as follows: $g: R^2 \to R^2 \times R^2$ by $g(x, y) = ((x, y), (x, y))$; $h: R^2 \times R^2 \to R \times R$ by $h((a, b), (c, d)) = (a + b, c - d)$; $k: R \times R \to R \times R$ by $k(u, v) = (u^2, v^2)$; $m: R \times R \to R$ by $m(x, y) = \frac{1}{4}(x - y)$. Prove that all these functions are continuous and that $xy = mkhg(x, y)$.

4 OPEN BALLS AND NEIGHBORHOODS

In the definition of continuity of a function f at a point a in a metric space (X, d), we are concerned with how f transforms those points $x \in X$ such that $d(x, a) < \delta$. If we give a name to this particular collection of points in X we shall be able to cast the definition of continuity in a more compact form.

Definition 4.1 Let (X, d) be a metric space. Let $a \in X$ and $\delta > 0$ be given. The subset of X consisting of those points $x \in X$ such that $d(a, x) < \delta$ is called the *open ball about a of radius δ* and is denoted by

$$B(a; \delta).$$

Thus, $x \in B(a; \delta)$ if and only if $x \in X$ and $d(x, a) < \delta$. Similarly, if (Y, d') is another metric space and $f: X \to Y$, we have $y \in B(f(a); \varepsilon)$ if and only if $y \in Y$ and $d'(y, f(a)) < \varepsilon$. Thus:

THEOREM 4.2 A function $f: (X, d) \to (Y, d')$ is continuous at a point $a \in X$ if and only if given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$f(B(a; \delta)) \subset B(f(a); \varepsilon).$$

For a function $f: X \to Y$ we have $f(U) \subset V$ if and only if $U \subset f^{-1}(V)$, where U and V are subsets of X and Y respectively. Therefore:

THEOREM 4.3 A function $f: (X, d) \to (Y, d')$ is continuous at a point $a \in X$ if and only if given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$B(a; \delta) \subset f^{-1}(B(f(a); \varepsilon)).$$

Given a point a in a metric space (X, d), the subset $B(a; \delta)$ of X, for each $\delta > 0$, is an example of the type of subset of X that is called a neighborhood of a.

DEFINITION 4.4 Let (X, d) be a metric space and $a \in X$. A subset N of X is called a *neighborhood of* a if there is a $\delta > 0$ such that

$$B(a; \delta) \subset N.$$

The collection $\mathfrak{N}_a$ of all neighborhoods of a point $a \in X$ is called a *complete system of neighborhoods* of the point a.

A neighborhood of a point $a \in X$ may be thought of as containing all the points of X that are sufficiently close to a or as "enclosing" a by virtue of the fact that it contains some open ball about a. In particular, for each $\delta > 0$, $B(a; \delta)$ is a neighborhood of a. These open balls have the property that they are neighborhoods of each of their points.

LEMMA 4.5 Let (X, d) be a metric space and $a \in X$. For each $\delta > 0$, the open ball $B(a; \delta)$ is a neighborhood of each of its points.

Proof. Let $b \in B(a; \delta)$. In order to show that $B(a; \delta)$ is a neighborhood of b we must show that there is an $\eta > 0$ such that $B(b; \eta) \subset B(a; \delta)$. Since $b \in B(a; \delta)$, $d(a, b) < \delta$. Choose $\eta < \delta - d(a, b)$. If $x \in B(b; \eta)$ then

$$d(a, x) \leqq d(a, b) + d(b, x) < d(a, b) + \eta < d(a, b) + \delta - d(a, b) = \delta,$$

and therefore $x \in B(a; \delta)$. Thus $B(b; \eta) \subset B(a; \delta)$ and $B(a; \delta)$ is a neighborhood of b.

We may describe this proof pictorially. We have started with an open ball $B(a; \delta)$ about a. We choose a point $b \in B(a; \delta)$. Then the minimum distance from b to points not in $B(a; \delta)$ is at least $\delta - d(a, b)$, as indicated in Figure 5, so that a ball about b

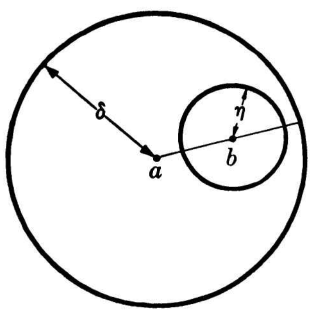


Figure 5

of radius $\eta < \delta - d(a, b)$ is contained in $B(a; \delta)$.

The complete system of neighborhoods of a point may be used to characterize continuity of a function at a point.

THEOREM 4.6 Let $f: (X, d) \to (Y, d')$. f is continuous at a point $a \in X$ if and only if for each neighborhood M of $f(a)$ there is a corresponding neighborhood N of a, such that

$$f(N) \subset M,$$

or equivalently,

$$N \subset f^{-1}(M).$$

Proof. First suppose that f is continuous at the point $a \in X$. We must show that, given a neighborhood M of $f(a)$, we can find a neighborhood N of a such that $f(N) \subset M$. Since M is a neighborhood of $f(a)$, there is an $\varepsilon > 0$ such that $B(f(a); \varepsilon) \subset M$. Since f is continuous at a, there is a $\delta > 0$ such that $f(B(a; \delta)) \subset B(f(a); \varepsilon)$. But $N = B(a; \delta)$ is a neighborhood of a, therefore

$$f(N) = f(B(a; \delta)) \subset B(f(a); \varepsilon) \subset M.$$

Conversely, suppose that f satisfies the property that for each neighborhood M of $f(a)$, there is a corresponding neighborhood N of a, such that $f(N) \subset M$. Let $\varepsilon > 0$ be given. To prove that f is continuous at a we must show that there is a $\delta > 0$ such that

$$f(B(a; \delta)) \subset B(f(a); \varepsilon).$$

But $B(f(a); \varepsilon) = M$ is a neighborhood of $f(a)$ and therefore there is a neighborhood N of a such that $f(N) \subset M$. Since N is a neighborhood of a, there is a $\delta > 0$ such that $B(a; \delta) \subset N$. Therefore

$$f(B(a; \delta)) \subset f(N) \subset M = B(f(a); \varepsilon).$$

The proof of the first part of the above theorem may be represented pictorially by considering an arbitrary neighborhood M of $f(a)$ (as indicated in Figure 6). Since M is a neigh-

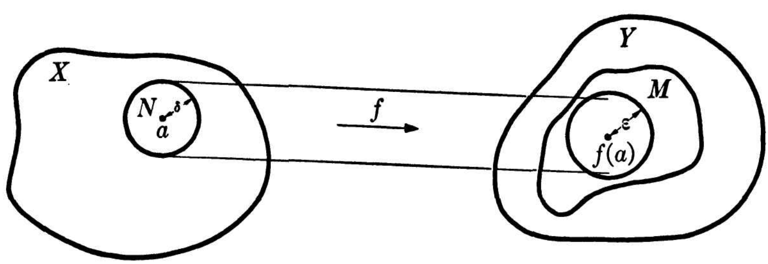


Figure 6

borhood of $f(a)$, it contains an open ball $B(f(a); \varepsilon)$ for some $\varepsilon > 0$. Since f is continuous at a, for some $\delta > 0$ the neighborhood $N = B(a; \delta)$ is carried into M by f. Similarly, the proof of the second part of the theorem may be depicted by Figure 7. We start with a neighborhood $M = B(f(a); \varepsilon)$ of $f(a)$. The assumed

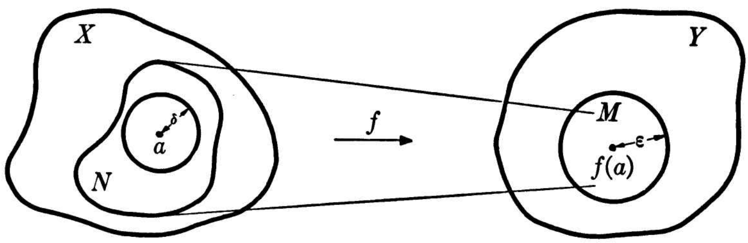


Figure 7

property of f allows us to assert that there is a neighborhood N of a that is carried into M by f. Since N is a neighborhood of a we have an open ball $B(a; \delta)$ contained in N, which must also be carried into M.

If N is a neighborhood of a point a in a metric space (X, d) and N' is a subset of X that contains N, then N' contains the same open ball about a that N does and therefore N' is also a neighborhood of a. Thus, the previous theorem becomes:

THEOREM 4.7 Let $f: (X, d) \to (Y, d')$. f is continuous at a point $a \in X$ if and only if for each neighborhood M of $f(a)$, $f^{-1}(M)$ is a neighborhood of a.

The collections of neighborhoods of points in a metric space possess five properties that will be of significance in the next chapter.

THEOREM 4.8 Let (X, d) be a metric space.

$N1$. For each point $a \in X$, there exists at least one neighborhood of a.

$N2$. For each point $a \in X$ and each neighborhood N of a, $a \in N$.

*N*3. For each point $a \in X$, if N is a neighborhood of a and $N' \supset N$, then N' is a neighborhood of a.

*N*4. For each point $a \in X$ and each pair N, M of neighborhoods of a, $N \cap M$ is also a neighborhood of a.

*N*5. For each point $a \in X$ and each neighborhood N of a, there exists a neighborhood O of a such that $O \subset N$ and O is a neighborhood of each of its points.

Proof. For $a \in X$, X is a neighborhood of a, thus $N1$ is true. $N2$ is trivial and $N3$ has already been discussed. To prove $N4$, let N and M be neighborhoods of a. Then N and M contain open balls $B(a; \delta_1)$ and $B(a; \delta_2)$ respectively and therefore $N \cap M$ contains the open ball $B(a; \delta)$, where $\delta = \text{minimum } \{\delta_1, \delta_2\}$. To prove $N5$, let N be a neighborhood of a. Then N contains an open ball $B(a; \delta)$ and by Lemma 4.5, $O = B(a; \delta)$ is a neighborhood of each of its points.

For a given point a in a metric space X, the collection of open balls with center a has been used to generate the complete system of neighborhoods at a, in the sense that the neighborhoods of a are precisely those subsets of X which contain one of these open balls.

DEFINITION 4.9 Let a be a point in a metric space X. A collection $\mathcal{B}_a$ of neighborhoods of a is called a *basis for the neighborhood system at a* if every neighborhood N of a contains some element B of $\mathcal{B}_a$.

As an example, if a is a point on the real line R, a basis for the neighborhood system at a is the collection of open intervals containing a.

EXERCISES

1. Let (X, d) be a metric space such that $d(x, y) = 1$ whenever $x \neq y$. Let $a \in X$. Prove that $\{a\}$ is a neighborhood of a and constitutes a basis for the system of neighborhoods at a. Prove that every subset of X is a neighborhood of each of its points.

2. Let $a \in R$ and $f: R \to R$ be defined by $f(x) = 0$ for $x \leqq a$, $f(x) = 1$ for $x > a$. Prove that f is not continuous at a, but is continuous at all other points.
3.. Let $f: X \to Y$ be a function from a metric space X into a metric space Y. Let $a \in X$ and let $\mathcal{B}_{f(a)}$ be a basis for the neighborhood system at $f(a)$. Prove that f is continuous at a if and only if for each $N \in \mathcal{B}_{f(a)}$, $f^{-1}(N)$ is a neighborhood of a.
4. Let a be a point on the real line R. Prove that each of the following collections of subsets of R constitute a basis for the system of neighborhoods at a:

   i) All closed intervals of the form $[a - \varepsilon, a + \varepsilon]$, $\varepsilon > 0$;
   ii) All open balls $B(a; \varepsilon)$, ε a positive rational number;
   iii) All open balls $B\left(a; \frac{1}{n}\right)$, n a positive integer;
   iv) All open balls $B\left(a; \frac{1}{n}\right)$, n a positive integer larger than some fixed integer k.

   Show that no finite collection of subsets of R can be a basis for the system of neighborhoods at a.
5. Let a be a point in a metric space X. Let N be the set of positive integers. Prove that there is a collection $\{B_n\}_{n \in N}$ of neighborhoods of a which constitutes a basis for the system of neighborhoods at a.
6. Let a and b be distinct points of a metric space X. Prove that there are neighborhoods N_a and N_b of a and b respectively such that $N_a \cap N_b = \emptyset$.
7. Let (X_1, d_1), (X_2, d_2), . . . , (X_n, d_n) be metric spaces and convert

$$X = \prod_{i=1}^{n} X_i$$

   into a metric space (X, d) in the standard manner. Prove that an open ball in (X, d) is the product of open balls from $X_1, X_2, \ldots, X_n$ respectively. Let $a_i \in X_i$, $i = 1, 2, \ldots, n$, and let $\mathcal{B}_{a_i}$ be a basis for the neighborhood system at a_i. Let $\mathcal{B}_a$ be the collection of all sets of the form $B_1 \times B_2 \times \ldots \times B_n$, $B_i \in \mathcal{B}_{a_i}$. Prove that $\mathcal{B}_a$ is a basis for the neighborhood system at $a = (a_1, a_2, \ldots, a_n) \in X$. Let $p_i: X \to X_i$, $i = 1, 2, \ldots, n$, be the projection that maps $p_i(a) = a_i$. Prove that each p_i is continuous. Let Y be a metric space and $f: Y \to X$ a function. Prove that f is continuous if and only if each of the n functions $p_i f$ is continuous.

8. Let R be the real numbers and $f: R \to R$ a continuous function. Suppose that for some number $a \in R$, $f(a) > 0$. Prove that there is a positive number k and a closed interval $F = [a - \delta, a + \delta]$ for some $\delta > 0$ such that $f(x) \geq k$ for $x \in F$.

5 LIMITS

The concept of limit of a sequence of real numbers may be generalized to an arbitrary metric space. First, let us recall the appropriate definition in the real line.

Definition 5.1 Let $a_1, a_2, \ldots$ be a sequence of real numbers. A real number a is said to be the *limit of the sequence* $a_1, a_2, \ldots$ if, given $\varepsilon > 0$, there is a positive integer N such that, whenever $n > N$, $|a - a_n| < \varepsilon$. In this event we shall also say that the sequence $a_1, a_2, \ldots$ *converges to* a and write $\lim_n a_n = a$.

Interpreting ε as an "arbitrary degree of closeness" and N as "sufficiently far out in the sequence," we see that we have defined $\lim_n a_n = a$ in the event that a_n may be made arbitrarily close to a by requiring that a_n be sufficiently far out in the sequence.

Now, suppose that we have a metric space (X, d) and a sequence $a_1, a_2, \ldots$ of points of X. Given a point $a \in X$ we measure the distance from a to the successive points of the sequence, by the sequence of real numbers $d(a, a_1), d(a, a_2), \ldots$. It is natural to say that the limit of the sequence $a_1, a_2, \ldots$ of points of X is the point a if the limit of the sequence of real numbers $d(a, a_1), d(a, a_2), \ldots$ is the real number 0.

Definition 5.2 Let (X, d) be a metric space. Let $a_1, a_2, \ldots$ be a sequence of points of X. A point $a \in X$ is said to be the *limit of the sequence* $a_1, a_2, \ldots$ if $\lim_n d(a, a_n) = 0$. Again, in this event, we shall say that the sequence

$a_1, a_2, \ldots$ *converges to* a and write $\lim_n a_n = a$.

COROLLARY 5.3 Let (X, d) be a metric space and $a_1, a_2, \ldots$ be a sequence of points of X. Then $\lim_n a_n = a$ for a point $a \in X$ if and only if for each neighborhood V of a there is an integer N such that $a_n \in V$ whenever $n > N$.

Proof. Let V be a neighborhood of a. For some $\varepsilon > 0$, $a \in B(a; \varepsilon) \subset V$. Thus if $\lim_n a_n = a$ there is an integer N such that whenever $n > N$, $d(a, a_n) < \varepsilon$ and hence $a_n \in V$. Conversely, given $\varepsilon > 0$, $B(a; \varepsilon)$ is a neighborhood of a. If there is an integer N such that for $n > N$, $a_n \in B(a; \varepsilon)$, then $d(a, a_n) < \varepsilon$ and $\lim_n a_n = a$.

If S is a set of infinite points, and there is at most a finite number of elements of S for which a certain statement is false, then the statement is said to be true for *almost all* the elements of S. Thus $\lim_n a_n = a$ if for each neighborhood V of a almost all the points a_n are in V.

Continuity may be characterized in terms of limits of sequences in accordance with the following theorem.

THEOREM 5.4 Let (X, d), (Y, d') be metric spaces. A function $f: X \to Y$ is continuous at a point $a \in X$ if and only if, whenever $\lim_n a_n = a$ for a sequence $a_1, a_2, \ldots$ of points of X, $\lim_n f(a_n) = f(a)$.

Proof. Suppose f is continuous at a and $\lim_n a_n = a$. Let V be a neighborhood of $f(a)$. Then $f^{-1}(V)$ is a neighborhood of a, so by Corollary 5.3 there is an integer N such that $a_n \in f^{-1}(V)$ whenever $n > N$. Consequently, $f(a_n) \in V$ whenever $n > N$. Thus, for each neighborhood V of $f(a)$ there is an integer N such that $f(a_n) \in V$ whenever $n > N$ and again, applying Corollary 5.3, $\lim_n f(a_n) = f(a)$.

To prove the "if" part of this theorem, we shall prove that if f is not continuous at a, then there is at least one sequence $a_1, a_2, \ldots$ of points of X, such that $\lim_n a_n = a$, but $\lim_n f(a_n) = f(a)$ is false. Since f is not continuous at a, there is a neighborhood V of $f(a)$ such that for each neighborhood U of a, $f(U) \not\subset V$. In particular, for each neighborhood $B\left(a; \frac{1}{n}\right)$, $n = 1, 2, \ldots$ $f\left(B\left(a; \frac{1}{n}\right)\right) \not\subset V$. Thus, for each positive integer n, there is a point a_n with $a_n \in B\left(a; \frac{1}{n}\right)$ and $f(a_n) \notin V$. Now $d(a, a_n) < \frac{1}{n}$ and therefore $\lim_n a_n = a$, whereas, $\lim_n f(a_n) = f(a)$ is impossible, since $f(a_n) \notin V$ for all n.

If $\lim_n a_n = a$, we can write $\lim_n f(a_n) = f(a)$ as $\lim_n f(a_n) = f(\lim_n a_n)$. We may therefore describe a continuous function as one that commutes with the operation of taking limits. It is worth noting that in proving f is continuous whenever f commutes with the operation of taking limits we have used the fact that the sequence of neighborhoods $B\left(a; \frac{1}{n}\right)$, n a positive integer, constitutes a basis for the neighborhood system at a.

In order to introduce the concept of distance from a point to a subset we shall recall some facts about the real number system.

DEFINITION 5.5 Let A be a set of real numbers. A number b is called an *upper bound* of A if $x \leqq b$ for each $x \in A$. A number c is called a *lower bound* of A if $c \leqq x$ for each $x \in A$. If A has both an upper and lower bound A is said to be *bounded.*

An upper bound b^* of A is called a *least upper bound* (abbreviated l.u.b.) of A if for each upper bound b of A, $b^* \leqq b$. A lower bound c^* of A is called a *greatest lower bound* (abbreviated g.l.b.) of A if for each lower bound c of A, $c \leqq c^*$.

Not every set of real numbers has an upper bound. One of the properties of the real number system, usually referred to as the *completeness postulate*, is that a non-empty set A of real numbers which has an upper bound necessarily has a l.u.b. Given a non-empty set B of real numbers which has a lower bound, the set of negatives of elements of B has an upper bound, hence a l.u.b. whose negative is a g.l.b. of B. Thus it follows that every non-empty set B of real numbers which has a lower bound has a g.l.b.

The greatest lower bound of a set A of real numbers may or may not be an element of A. For example, 0 is a g.l.b. of $[0, 1]$ and $0 \in [0, 1]$, whereas 0 is also a g.l.b. of $(0, 1)$ but $0 \notin (0, 1)$. In any event, the g.l.b. of a set of real numbers must be arbitrarily close to that set.

LEMMA 5.6 Let b be a greatest lower bound of the non-empty subset A. Then, for each $\varepsilon > 0$, there is an element $x \in A$ such that

$$x - b < \varepsilon.$$

Proof. Suppose there were an $\varepsilon > 0$ such that $x - b \geqq \varepsilon$ for each $x \in A$. Then $b + \varepsilon \leqq x$ for each $x \in A$ and $b + \varepsilon$ would be a lower bound of A. Since b is a g.l.b. of A, we obtain the contradiction $b + \varepsilon \leqq b$.

COROLLARY 5.7 Let b be a greatest lower bound of the non-empty subset A of real numbers. Then there is a sequence $a_1, a_2, \ldots$ of real numbers such that $a_n \in A$ for each n and $\lim_n a_n = b$.

Proof. For $\varepsilon = \frac{1}{n}$ we obtain an element $a_n \in A$ such that $a_n - b < \frac{1}{n}$. Since b is a lower bound of A, $0 \leqq a_n - b$. Therefore $\lim_n a_n = b$.

DEFINITION 5.8 Let (X, d) be a metric space. Let $a \in X$ and let A be a non-empty subset of X. The greatest lower bound of the set of numbers of the form $d(a, x)$ for $x \in A$ is called the *distance between a and A* and is denoted by $d(a, A)$.

From Corollary 5.7 we obtain:

COROLLARY 5.9 Let (X, d) be a metric space, $a \in X$, and A a non-empty subset of X. Then there is a sequence $a_1, a_2, \ldots$ of points of A such that $\lim_n d(a, a_n) = d(a, A)$.

EXERCISES

1. Let $X_1, X_2, \ldots, X_k$ be metric spaces and convert $X = \prod_{i=1}^{k} X_i$ into a metric space in the standard manner. Each of the points $a_1, a_2, \ldots$ of a sequence of points of X has k coordinates; that is $a_n = (a_1^n, a_2^n, \ldots, a_k^n) \in X$, $n = 1, 2, \ldots$. Let $c = (c_1, c_2, \ldots, c_k) \in X$. Prove that $\lim_n a_n = c$ if and only if $\lim_n a_i^n = c_i$, $i = 1, 2, \ldots, k$.
2. In each of the three metric spaces (R^k, d), (R^k, d'), (R^k, d'') considered in Section 2, prove that limits of sequences are the same.
3. Prove that a subsequence of a convergent sequence is convergent and converges to the same limit as the original sequence.
4. A sequence of real numbers $a_1, a_2, \ldots$ is called *monotone non-decreasing* if $a_i \leqq a_{i+1}$ for each i and called *monotone non-increasing* if $a_i \geqq a_{i+1}$ for each i. A sequence which is either monotone non-decreasing or monotone non-increasing is said to be *monotone.* The sequence is said to be *bounded above* if there is a number K such that $a_i \leqq K$ for each i and *bounded below* if there is a number M such that $a_i \geqq M$ for each i. A sequence which is both bounded above and bounded below is called *bounded.* Prove that a convergent sequence of real numbers is bounded. Prove that a monotone non-decreasing sequence of real numbers which is bounded above converges to a limit a and that a is the l.u.b. of the set $\{a_1, a_2, \ldots\}$. Similarly prove that a monotone non-increasing sequence which is bounded below converges to a limit b and that b is the g.l.b. of the set $\{a_1, a_2, \ldots\}$.
5. Let $a_1, a_2, \ldots$ be a bounded sequence of real numbers. Since each of the sets $A_n = \{a_n, a_{n+1}, \ldots\}$ is bounded we may set $v_n =$ g.l.b. A_n, $u_n =$ l.u.b. A_n. Observe that $v_n \leqq u_n$; $v_1, v_2, \ldots$ is monotone non-decreasing and bounded above; and $u_1, u_2, \ldots$ is monotone non-increasing and bounded below. Let $V = \lim_n v_n$ and $U = \lim_n u_n$. Prove that there are subsequences of $a_1, a_2, \ldots$ which converge to U and V respectively (thus a bounded sequence of real numbers has a convergent subsequence). Prove that $a_1, a_2, \ldots$ converges if and only if $U = V$.

6. Let (X, d) be a metric space and A a non-empty subset of X. For $x, y \in X$, prove that $d(x, A) \leqq d(x, y) + d(y, A)$.
7. Let A be a non-empty subset of a metric space (X, d). Define the function $f: X \rightarrow R$ by $f(x) = d(x, A)$. Prove that f is continuous.
8. Let A be a non-empty subset of a metric space (X, d) and let $x \in X$. Prove that $d(x, A) = 0$ if and only if every neighborhood of x contains a point of A.
9. Let (X, d) be a metric space. Define a distance function d^* on $X \times X$ by the method of Theorem 2.3. Prove that the function $d: (X \times X, d^*) \rightarrow (R, d)$ is continuous.

6 OPEN SETS AND CLOSED SETS

In a metric space, the open ball $B(a; \delta)$ is a neighborhood of each of its points (Lemma 4.5). The collection of subsets possessing this property plays a fundamental role in topology.

Definition 6.1 A subset O of a metric space is said to be *open* if O is a neighborhood of each of its points.

Open sets may be characterized directly in terms of open balls.

Theorem 6.2 A subset O of a metric space (X, d) is an open set if and only if it is a union of open balls.

Proof. Suppose O is open. Then for each $a \in O$, there is an open ball $B(a; \delta_a) \subset O$. Therefore $O = \bigcup_{a \in O} B(a; \delta_a)$ is a union of open balls. Conversely, if O is a union of open balls, then using the centers of these balls as the elements of an indexing set we can write $O = \bigcup_{a \in I} B(a; \delta_a)$. If $x \in O$, then $x \in B(a; \delta_a)$ for some $a \in I$. $B(a; \delta_a)$ is a neighborhood of x and since $B(a; \delta_a) \subset O$, by $N3$, O is a neighborhood of x. Thus O is a neighborhood of each of its points, and by Definition 6.1, O is open.

Most of the functions considered in topology are continuous. Open sets provide a simple characterization of continuity.

Theorem 6.3 Let $f: (X, d) \to (Y, d')$. Then f is continuous if and only if for each open set O of Y, the subset $f^{-1}(O)$ is an open subset of X.

Proof. First, suppose f is continuous. Let $O \subset Y$ be open. We must show that $f^{-1}(O)$ is open; that is, $f^{-1}(O)$ is a neighborhood of each of its points. To this end, let $a \in f^{-1}(O)$, then $f(a) \in O$ and O is a neighborhood of $f(a)$. Since f is continuous at a, Theorem 4.7 may be applied, yielding $f^{-1}(O)$ is a neighborhood of a.

Conversely, suppose for each open set $O \subset Y, f^{-1}(O)$ is open. Let $a \in X$ and let M be a neighborhood of $f(a)$. Then there is an $\varepsilon > 0$ such that $B(f(a); \varepsilon) \subset M$. But $B(f(a); \varepsilon)$ is open and therefore $f^{-1}(B(f(a); \varepsilon))$ is open. Since $a \in f^{-1}(B(f(a); \varepsilon))$, this subset is a neighborhood of a. Therefore $f^{-1}(M)$ contains a neighborhood of a and f is continuous at a. Since a was arbitrary, f is continuous.

Just as the collections of neighborhoods of points in a metric space possess certain significant properties so do the collection of open sets in a metric space.

Theorem 6.4 Let (X, d) be a metric space.

*O*1. The empty set is open.

*O*2. X is open.

*O*3. If $O_1, O_2, \ldots, O_n$ are open, then $O_1 \cap O_2 \cap \ldots \cap O_n$ is open.

*O*4. If for each $\alpha \in I$, O_α is an open set, then $\bigcup_{\alpha \in I} O_\alpha$ is open.

Proof. The empty set is open, for in order for it not to be open there would have to be a point $x \in \emptyset$. Given a point $a \in X$, for any $\delta > 0$, $B(a; \delta) \subset X$ and therefore X is a neighborhood of each of its points; that is, X is open. To prove *O*3, let $a \in O_1 \cap O_2 \cap \ldots \cap O_n$, where for $i = 1, 2, \ldots, n$, O_i is open. Then each O_i is a neighborhood of a. By $N4$, the intersection of two neighborhoods of a is again a neighborhood of a, and hence by induction, the intersection of a finite number of neighborhoods of a is again a neighborhood of a. Therefore $O_1 \cap O_2 \cap \ldots \cap O_n$ is a neighborhood of each of its points. Finally, to prove $O4$, let $a \in O = \bigcup_{\alpha \in I} O_\alpha$, where for each $\alpha \in I$, O_α is open. Then $a \in O_\beta$ for some $\beta \in I$ and O_β is a neighborhood of a. Since $O_\beta \subset O$, by $N3$, O is a neighborhood of a. Therefore O is a neighborhood of each of its points.

Definition 6.5 A subset F of a metric space is said to be *closed* if its complement, $C(F)$, is open.

In the real number system, a closed interval $[a, b]$ is a closed set, for its complement is the union of the two open sets O_1 and O_2, where O_1 is the set of real numbers x such that $x < a$ and O_2 is the set of real numbers x such that $x > b$. A common mistake is the assumption that a set cannot be both open and closed. In any metric space (X, d), the two sets $\emptyset$ and X are open, and therefore their complements X and $\emptyset$ are closed. Thus, X and also $\emptyset$ are both open and both closed. Whether or not, in a given metric space, there are other subsets that are simultaneously open and closed, is a significant topological property, which we shall subsequently describe by the adjective "connected." In any event, the adjectives *open* and *closed* are not mutually exclusive. Nor, for that matter, are they all-inclusive, for we shall shortly give an example of a subset of the real number system that is neither open nor closed.

Definition 6.6 Let A be a subset of a metric space X. A point $b \in X$ is called a *limit point* of A if every neighborhood of b contains a point of A different from b.

If b is a limit point of A then each of the open balls $B\left(b; \frac{1}{n}\right)$ contains a point $a_n \in A$ and $\lim_n a_n = b$. Thus a limit point of a set is the limit of a convergent sequence of points of A. The converse is false, for the point b may be a point of A while for some δ, $B(b; \delta)$ contains no point of A other than b. Thus b is not a limit point of A although the sequence $b, b, \ldots$ converges to b. In this latter case b is called an *isolated point* of A.

THEOREM 6.7 In a metric space X, a set $F \subset X$ is closed if and only if F contains all its limit points.

Proof. Let F' denote the set of limit points of F. First suppose F is closed and consequently $C(F)$ is open. Choose a point $b \notin F$. Since $C(F)$ is open there is a $\delta > 0$ such that $B(b; \delta) \subset C(F)$ or $B(b; \delta) \cap F = \emptyset$. Hence $b \notin F'$ and $F' \subset F$.

Conversely, suppose $F' \subset F$, or equivalently, $C(F) \subset C(F')$. If $b \in C(F)$, then $b \notin F'$. It follows that for some $\delta > 0$, $B(b; \delta) \cap F = \emptyset$, or $B(b; \delta) \subset C(F)$. Hence $C(F)$ is open and F is closed.

THEOREM 6.8 In a metric space (X, d), a set $F \subset X$ is closed if and only if for each sequence $a_1, a_2, \ldots$ of points of F that converges to a point $a \in X$ we have $a \in F$.

Proof. First, let F be closed. Suppose $\lim_n a_n = a$ and $a_n \in F$ for $n = 1, 2, \ldots$. If the set of points $\{a_1, a_2, \ldots\}$ is infinite then every neighborhood of a contains infinitely many points of F, a is a limit point of F, and so by Theorem 6.7, $a \in F$. If this set of points is finite, then for some integer N, $a_n = a_m$ whenever $n, m > N$. Since $\lim_n a_n = a$, $d(a_n, a) = 0$ for $n > N$ or $a_n = a$, whence $a \in F$. Conversely, suppose that F is a set such that for each sequence with $\lim_n a_n = a$ and $a_n \in F$ for all n, we have $a \in F$. If b is a limit point of F then b is the limit of a convergent sequence of points of F and $b \in F$. Thus by Theorem 6.7 F is closed.

Finally, we may characterize closed sets in terms of distance from a point to a set.

THEOREM 6.9 A subset F of a metric space (X, d) is closed if and only if for each point $x \in X$, $d(x, F) = 0$ implies $x \in F$.

Proof. First, suppose F is closed. Let $x \in X$ be such that $d(x, F) = 0$. By Corollary 5.9 there is a sequence of points of F such that $\lim_n d(x, a_n) = 0$. Thus, every neighborhood of x contains points of F. If some $a_n = x$, x is in F. Otherwise each a_n is different from x, so that x is a limit point of the sequence and hence of F. Thus, by Theorem 6.7, $x \in F$. Conversely, suppose that F is such that $d(x, F) = 0$ implies $x \in F$. If x is a limit point of F then $d(x, F) = 0$. Thus in this case F contains all its limit points and is closed.

Continuity may be characterized by means of closed sets.

THEOREM 6.10 Let (X, d), (Y, d') be metric spaces. A function $f: X \to Y$ is continuous if and only if for each closed subset A of Y, the set $f^{-1}(A)$ is a closed subset of X.

Proof. For $A \subset Y$, we have $C(f^{-1}(A)) = f^{-1}(C(A))$. But f is continuous if and only if the inverse image of each open set is an open set, and this is true if and only if the inverse image of each closed set is a closed set.

As a final result in this section we record the following facts about closed sets.

THEOREM 6.11 Let (X, d) be a metric space.

*C*1. X is closed.

*C*2. $\emptyset$ is closed.

*C*3. The union of a finite collection of closed sets is closed.

*C*4. The intersection of a family of closed sets is closed.

Proof. *C*1 and *C*2 have already been discussed. *C*3 and *C*4 follow from the application of DeMorgan's formulas to the corresponding properties *O*3 and *O*4 of open sets.

The union of closed sets need not, in general, be a closed set, as may be seen by the following example. For each positive integer n let F_n be the closed interval $\left[\frac{1}{n}, 1\right]$. Then $\bigcup_{n=1}^{\infty} F_n = (0, 1]$, where $(0, 1]$ is the set of real numbers x such that $0 < x \leqq 1$. The set $(0, 1]$ is not closed, for 0 is a limit point of the set but is not in the set.

EXERCISES

1. Let (X_i, d_i), $i = 1, 2, \ldots, n$ be metric spaces. Let $X = \prod_{i=1}^{n} X_i$ and let (X, d) be the metric space defined in the standard manner by Theorem 2.3. For $i = 1, 2, \ldots, n$, let O_i be an open subset of X_i. Prove that the subset $O_1 \times O_2 \times \ldots \times O_n$ of X is open and that each open subset of X is a union of sets of this form. [A collection of open sets of a metric space is called a *basis for the open sets* if each open set is a union of sets in this collection. For example, the open balls in a metric space form a basis for the open sets.]
2. Let X be a set and d the distance function on X defined by $d(x, x) = 0$, $d(x, y) = 1$ for $x \neq y$. Prove that each subset of (X, d) is open.
3. Let (X, d_1), (Y, d_2) be metric spaces. Let $f: X \to Y$ be continuous. Define a distance function d on $X \times Y$ in the standard manner. Prove that the graph Γ_f of f is a closed subset of $(X \times Y, d)$.
4. Let $f: R \to R$ be defined by

$$f(x) = \frac{1}{x}, \; x > 0,$$

$$f(x) = 0, \; x \leqq 0.$$

   Prove that the graph Γ_f is a closed subset of (R^2, d), but that f is not continuous.
5. Let A be a closed, non-empty subset of the real numbers that has a lower bound. Prove that A contains its greatest lower bound.
6. Let A be a subset of a metric space. Let A' be the set of limit points of A and A^i the set of isolated points of A. Prove that $A' \cap A^i = \emptyset$ and $A \subset A' \cup A^i$. The set $\bar{A} = A' \cup A^i$ is called the *closure* of A. Prove that $x \in \bar{A}$ if and only if there is a sequence of points of A

which converges to x. Prove that if F is a closed set such that $A \subset F$ then $\bar{A} \subset F$. Prove that $\bar{A}$ is the intersection of all such closed sets F and hence is closed.

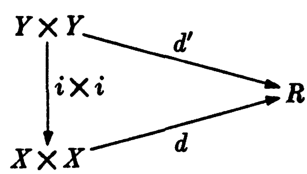
7 SUBSPACES AND EQUIVALENCE OF METRIC SPACES

Let (X, d) be a metric space. Given a non-empty subset Y of X we may convert Y into a metric space by restricting the distance function d to $Y \times Y$. In this manner each non-empty subset Y of X gives rise to a new metric space $(Y, d \mid Y \times Y)$. On the other hand, we may be given two metric spaces (X, d) and (Y, d'). If $Y \subset X$, it makes sense to ask whether or not d' is the restriction of d.

DEFINITION 7.1 Let (X, d) and (Y, d') be metric spaces. We say that (Y, d') is a *subspace* of (X, d) if:

1. $Y \subset X$;
2. $d' = d \mid Y \times Y$.

Let $Y \subset X$ and $i: Y \to X$ be an inclusion mapping. Denote by $i \times i: Y \times Y \to X \times X$ the inclusion mapping defined by $(i \times i)(y_1, y_2) = (y_1, y_2)$. Then (Y, d') is a subspace of (X, d) if the diagram



is commutative. There are as many subspaces of a metric space

(X, d) as there are non-empty subsets of X.

Example 1 Let Q be the set of rational numbers. Define $d_Q: Q \times Q \to R$ by $d_Q(a, b) = |a - b|$. Then (Q, d_Q) is a subspace of (R, d).

Example 2 Let I^n (the unit n-cube) be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers such that $0 \leqq x_i \leqq 1$, for $i = 1, 2, \ldots, n$. Define $d_c: I^n \times I^n \to R$ by $d_c((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \underset{1 \leqq i \leqq n}{\text{maximum}} \{|x_i - y_i|\}$. Then (I^n, d_c) is a subspace of (R^n, d).

Example 3 Let S^n (the n-sphere) be the set of all $(n + 1)$-tuples $(x_1, x_2, \ldots, x_{n+1})$ of real numbers such that $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1$. Define $d_S: S^n \times S^n \to R$ by

$$d_S((x_1, x_2, \ldots, x_{n+1}), (y_1, y_2, \ldots, y_{n+1})) = \sqrt{\sum_{i=1}^{n+1} (x_i - y_i)^2}.$$

Then (S^n, d_S) is a subspace of the Euclidean space (R^{n+1}, d').

Example 4 Let A be the set of all $(n + 1)$-tuples $(x_1, x_2, \ldots, x_{n+1})$ of real numbers such that $x_{n+1} = 0$. Define $d_A: A \times A \to R$ by

$$d_A((x_1, x_2, \ldots, x_n, 0), (y_1, y_2, \ldots, y_n, 0)) = \underset{1 \leqq i \leqq n}{\text{maximum}} \{|x_i - y_i|\}.$$

Then (A, d_A) is a subspace of (R^{n+1}, d).

Theorem 7.2 Let (Y, d') be a subspace of (X, d). Then the inclusion mapping $i: Y \to X$ is continuous.

Proof. Given $a \in Y$ and $\varepsilon > 0$, choose $\delta = \varepsilon$. If $d'(a, y) < \delta$, then $d(i(a), i(y)) = d(a, y) = d'(a, y) < \delta = \varepsilon$.

The metric space (A, d_A) of Example 4 is in most respects a copy of the metric space (R^n, d). The only distinction between (R^n, d) and (A, d_A) is that a point of R^n is an n-tuple of real numbers, whereas a point of A is an $(n + 1)$-tuple of real numbers of which the last one is zero. The relationship between the metric spaces (R^n, d) and (A, d_A) is an example of the relationship called "metric equivalence" or "isometry."

Definition 7.3 Two metric spaces (A, d_A) and (B, d_B) are said to be *metrically equivalent* or *isometric* if there are inverse functions $f:A \to B$ and $g:B \to A$ such that, for each $x, y \in A$, $d_B(f(x), f(y)) = d_A(x, y)$, and for each $u, v \in B$, $d_A(g(u), g(v)) = d_B(u, v)$. In this event we shall say that the *metric equivalence* or *isometry is defined by f and g.*

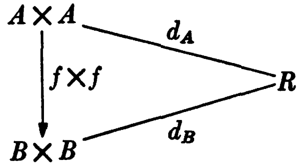
Theorem 7.4 A necessary and sufficient condition that two metric spaces (A, d_A) and (B, d_B) be metrically equivalent is that there exist a function $f:A \to B$ such that:

1. f is one-one;
2. f is onto;
3. for each $x, y \in A$, $d_B(f(x), f(y)) = d_A(x, y)$.

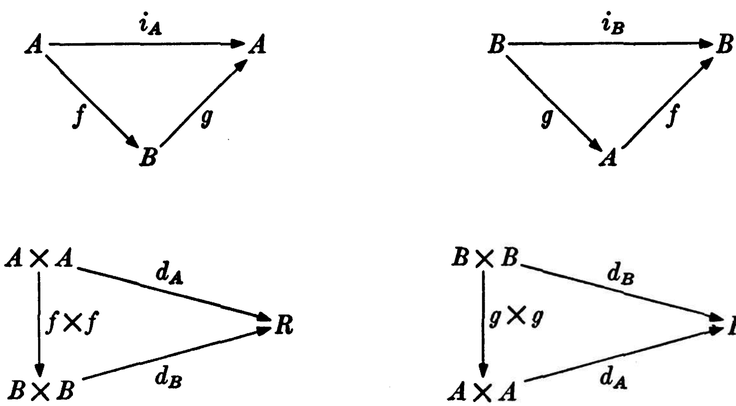
Proof. The stated conditions are necessary, for if (A, d_A) and (B, d_B) are metrically equivalent, there are inverse functions $f:A \to B$ and $g:B \to A$, and therefore f is one-one and onto. Conversely, suppose a function $f:A \to B$ with the stated properties exists. Then f is invertible and the function $g:B \to A$ such that f and g are inverse functions is determined by setting $g(b) = a$ if $f(a) = b$. For $u, v \in B$, let $x = g(u)$, $y = g(v)$. Then

$$d_A(g(u), g(v)) = d_A(x, y) = d_B(f(x), f(y)) = d_B(u, v).$$

Given metric spaces (A, d_A) and (B, d_B) and functions $f:A \to B$ and $g:B \to A$, let us denote by $f \times f:A \times A \to B \times B$ the function defined by setting $(f \times f)(x, y) = (f(x), f(y))$ for $x, y \in A$ and, similarly, let $g \times g:B \times B \to A \times A$ be defined by setting $(g \times g)(u, v) = (g(u), g(v))$ for $u, v \in B$. The statement that $d_B(f(x), f(y)) = d_A(x, y)$ for $x, y \in A$ is equivalent to the statement that the diagram



is commutative (one may also describe this relation by saying that the function $f: A \to B$ is "distance preserving"). In terms of diagrams, the statement that (A, d_A) and (B, d_B) are metrically equivalent is the statement that there exist functions $f: A \to B$, $g: B \to A$ such that the four diagrams



are commutative (where $i_A: A \to A$ and $i_B: B \to B$ are identity mappings). The first two diagrams express the fact that f and g are inverse functions and the last two diagrams express the fact that f and g "preserve distances." Since the distance between x and y in A is the same as the distance between $f(x)$ and $f(y)$ in B, f is continuous. Similarly, g is continuous. Thus:

LEMMA 7.5 Let a metric equivalence between (A, d_A) and (B, d_B) be defined by inverse functions $f:A \to B$ and $g:B \to A$. Then both f and g are continuous.

From the point of view of considerations that relate only to the concept of continuity, the relationship of metric equivalence is too narrow. We are led to define a broader concept of equivalence in which we drop the requirement of "preservation of distance"; that is, the commutativity of the last pair of diagrams, and merely require that the first two diagrams be commutative and the functions in these diagrams be continuous.

DEFINITION 7.6 Two metric spaces (A, d_A) and (B, d_B) are said to be *topologically equivalent* if there are inverse functions $f:A \to B$ and $g:B \to A$ such that f and g are continuous. In this event we say that the *topological equivalence is defined by f and g.*

As a corollary to Lemma 7.5 we obtain:

COROLLARY 7.7 Two metric spaces that are metrically equivalent are topologically equivalent.

The converse of this corollary is false; that is, there are metric spaces that are topologically equivalent, but are not metrically equivalent. For example, a circle of radius 1 is topologically equivalent to a circle of radius 2 (considered as subspaces of (R^2, d)), but the two are not metrically equivalent.

The following two results furnish a sufficient condition for the topological equivalence of two metric spaces with the same underlying sets.

LEMMA 7.8 Let (X, d_1) and (X, d_2) be two metric spaces. If there exists a number $K > 0$ such that for each $x, y \in X$, $d_2(x, y) \leqq K d_1(x, y)$, then the identity mapping

$$i:(X, d_1) \to (X, d_2)$$

is continuous.

Proof. Given $\varepsilon > 0$ and $a \in X$, set $\delta = \varepsilon/K$. If $d_1(x, a) < \delta$ then $d_2(i(x), i(a)) = d_2(x, a) \leqq K \cdot d_1(x, a) < K\delta = \varepsilon$.

Corollary 7.9 Let (X, d) and (X, d') be two metric spaces with the same underlying set. If there exist positive numbers K and K' such that for each $x, y \in X$,

$$d'(x, y) \leqq K \cdot d(x, y),$$
$$d(x, y) \leqq K' \cdot d'(x, y),$$

then the identity mappings define a topological equivalence between (X, d) and (X, d').

We have discussed the two metric spaces (R^n, d) and (R^n, d'), where the distance function d is determined by the maximum distance between coordinates, and the distance function d' is what is called the Euclidean distance function. For each pair of points $x, y \in R^n$, the inequality $d(x, y) \leqq d'(x, y) \leqq \sqrt{n}\, d(x, y)$ holds. It therefore follows from Corollary 7.9 that the metric spaces (R^n, d) and (R^n, d') are topologically equivalent.

Theorem 7.10 Let (X, d) and (Y, d') be two metric spaces. Let $f: X \to Y$ and $g: Y \to X$ be inverse functions. Then the following four statements are equivalent:

1. f and g are continuous;
2. A subset O of X is open if and only if $f(O)$ is an open subset of Y;
3. A subset F of X is closed if and only if $f(F)$ is a closed subset of Y;
4. For each $a \in X$ and subset N of X, N is a neighborhood of a if and only if $f(N)$ is a neighborhood of $f(a)$.

Proof. $1 \Rightarrow 2$. Let O be an open subset of X. Then $f(O) = g^{-1}(O)$ is open since g is continuous. Conversely, if $f(O)$ is an open subset of Y, then $f^{-1}(f(O)) = O$ is open since f is continuous.

$2 \Rightarrow 4$. For each $a \in X$ and $N \subset X$, N is a neighborhood of a if and only if N contains an open set O

containing a if and only if $f(N)$ contains an open set $O' = f(O)$ containing $f(a)$ if and only if $f(N)$ is a neighborhood of $f(a)$.

$4 \Rightarrow 1$. Let $a \in X$ and let U be a neighborhood of $f(a)$. Then $f^{-1}(U)$ is a neighborhood of a, for $U = f(f^{-1}(U))$ is a neighborhood of $f(a)$. Thus f is continuous. Similarly, let $b \in Y$ and let V be a neighborhood of $g(b)$. Then $g^{-1}(V) = f(V)$ is a neighborhood of $f(g(b)) = b$, and g is continuous.

Thus, statements 1, 2, and 4 are equivalent. We leave it to the reader to verify that statements 2 and 3 are equivalent.

Statement 1 in Theorem 7.10 is, of course, the statement that the metric spaces (X, d) and (Y, d') are topologically equivalent. Consequently, Theorem 7.10 asserts that two metric spaces are topologically equivalent if and only if there exist inverse functions that establish either a one-one correspondence between the open sets of the two spaces, a one-one correspondence between the closed sets of the two spaces, or a one-one correspondence between the complete systems of neighborhoods of the two spaces.

Both metrically equivalent and topologically equivalent are equivalence relations defined on a collection of metric spaces. By Corollary 7.7, each equivalence class of metrically equivalent metric spaces is contained in an equivalence class of topologically equivalent metric spaces. Distinguishing which topologically equivalent equivalence class a metric space belongs to is a coarser, but consequently more fundamental, distinction. By Theorem 7.10, this is determined by the collection of open sets, or the "topology" of the space.

EXERCISES

1. For each pair of points $a, b \in R^n$, prove that there is a topological equivalence between (R^n, d) and itself defined by inverse functions $f: R^n \to R^n$ and $g: R^n \to R^n$ such that $f(a) = b$. [*Hint:* If $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$, define f by setting

$f(x_1, x_2, \ldots, x_n) = (x_1 + b_1 - a_1, x_2 + b_2 - a_2, \ldots, x_n + b_n - a_n)$.]

2. Prove that the open interval $(-\pi/2, \pi/2)$, considered as a subspace of the real number system, is topologically equivalent to the real number system. Prove that any two open intervals, considered as subspaces of the real number system, are topologically equivalent. Prove that any open interval, considered as a subspace of the real number system, is topologically equivalent to the real number system.
3. For $i = 1, 2, \ldots, n$, let the metric space (X_i, d_i) be topologically equivalent to the metric space (Y_i, d'_i). Prove that if

$$X = \prod_{i=1}^{n} X_i \quad \text{and} \quad Y = \prod_{i=1}^{n} Y_i$$

are converted into metric spaces in the standard manner, then these two metric spaces are topologically equivalent.

4. The open n-cube is the set of all points $x = (x_1, x_2, \ldots, x_n) \in R^n$ such that $0 < x_i < 1$ for $i = 1, 2, \ldots, n$. Prove that the open n-cube, considered as a subspace of (R^n, d), is topologically equivalent to (R^n, d). [*Hint:* Use the results of Problems 2 and 3.]
5. Let XRY mean that the metric space X is isometric to the metric space Y. Prove that: (i) XRX; (ii) if XRY then YRX; and (iii) if XRY and YRZ then XRZ. Do the same if XRY means that the metric space X is topologically equivalent to the metric space Y.
6. Let (Y, d') be a subspace of the metric space (X, d). Prove that a subset $O' \subset Y$ is an open subset of (Y, d') if and only if there is an open subset O of (X, d) such that $O' = Y \cap O$. Prove that a subset $F' \subset Y$ is a closed subset of (Y, d') if and only if there is a closed subset F of (X, d) such that $F' = Y \cap F$. For a point $a \in Y$, prove that a subset $N' \subset Y$ is a neighborhood of a if and only if there is a neighborhood N of a in (X, d) such that $N' = Y \cap N$.
7. Let (Y, d') be a subspace of (X, d). Let $a_1, a_2, \ldots$ be a sequence of points of Y and let $a \in Y$. Prove that if $\lim_n a_n = a$ in (Y, d'), then $\lim_n a_n = a$ in (X, d). [The converse is false unless one assumes that all the points mentioned lie in Y; see the next problem.]
8. Consider the subspace (Q, d_Q) (the rational numbers) of (R, d). Let $a_1, a_2, \ldots$ be a sequence of rational numbers such that $\lim_n a_n = \sqrt{2}$. Prove that, given $\varepsilon > 0$, there is a positive integer N such that for $n, m > N$, $|a_n - a_m| < \varepsilon$. Does the sequence $a_1, a_2, \ldots$ converge when considered to be a sequence of points of (Q, d_Q)?

8 AN INFINITE DIMENSIONAL EUCLIDEAN SPACE

In this section we shall define a metric space H, sometimes called Hilbert space, which contains as subspaces isometric copies of the various Euclidean spaces (R^n, d'). A point u of H is a sequence $u_1, u_2, \ldots$ of real numbers such that the series $\sum_{i=1}^{\infty} u_i^2$ is convergent. Let $u = (u_1, u_2, \ldots)$ and $v = (v_1, v_2, \ldots)$ be in H. Our intention is to define a metric on H by setting

$$d(u, v) = \left[\sum_{i=1}^{\infty} (u_i - v_i)^2\right]^{1/2}.$$

In order to do this we must first know that the series in brackets converges. To accomplish this we shall make use of the following result, which is frequently referred to as Schwarz's lemma or Cauchy's inequality.

LEMMA 8.1 Let $(u_1, u_2, \ldots, u_n)$, $(v_1, v_2, \ldots, v_n)$ be n-tuples of real numbers, then

$$\sum_{i=1}^{n} u_i v_i \leqq \left[\sum_{i=1}^{n} u_i^2\right]^{1/2} \left[\sum_{i=1}^{n} v_i^2\right]^{1/2}.$$

Proof. It suffices to prove that

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leqq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right).$$

To this end, we consider, for an arbitrary real number λ, the expression $\sum_{i=1}^{n} (u_i + \lambda v_i)^2$. We have,

$$0 \leqq \sum_{i=1}^{n} (u_i + \lambda v_i)^2 = \sum_{i=1}^{n} u_i^2 + 2\lambda \sum_{i=1}^{n} u_i v_i + \lambda^2 \sum_{i=1}^{n} v_i^2.$$

Therefore, the quadratic equation in λ,

$$0 = \sum_{i=1}^{n} u_i^2 + 2\lambda \sum_{i=1}^{n} u_i v_i + \lambda^2 \sum_{i=1}^{n} v_i^2,$$

can have at most one real solution. Consequently,

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 - \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right) \leqq 0,$$

or

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leqq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right).$$

COROLLARY 8.2 Let $u = (u_1, u_2, \ldots)$, $v = (v_1, v_2, \ldots)$ be in H with $U = \sum_{i=1}^{\infty} u_i^2$, $V = \sum_{i=1}^{\infty} v_i^2$. Then the series $\sum_{i=1}^{\infty} u_i v_i$ is absolutely convergent and $\sum_{i=1}^{\infty} |u_i v_i| \leqq U^{1/2}V^{1/2}$.

Proof. For each positive integer n

$$\sum_{i=1}^{n} |u_i v_i| = \sum_{i=1}^{n} |u_i|\,|v_i| \leqq \left[\sum_{i=1}^{n} |u_i|^2\right]^{1/2}\left[\sum_{i=1}^{n} |v_i|^2\right]^{1/2} \leqq U^{1/2}V^{1/2}.$$

Thus the partial sums of this series of positive terms are bounded and the series converges to a limit not greater than $U^{1/2}V^{1/2}$.

Furthermore, if α and β are real numbers and we set $\alpha u + \beta v = (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, \ldots)$ then $\alpha u + \beta v$ is also in H for $\sum_{i=1}^{\infty} (\alpha u_i + \beta v_i)^2$ is the sum of three absolutely convergent series. In particular $u + v \in H$ and

$$\sum_{i=1}^{\infty} (u_i + v_i)^2 = \sum_{i=1}^{\infty} |u_i^2 + 2u_i v_i + v_i^2| \leqq \sum_{i=1}^{\infty} u_i^2 + 2\sum_{i=1}^{\infty} |u_i v_i| + \sum_{i=1}^{\infty} v_i^2$$
$$\leqq U + 2U^{1/2}V^{1/2} + V = (U^{1/2} + V^{1/2})^2.$$

Taking square roots we obtain

COROLLARY 8.3 $\left[\sum_{i=1}^{\infty} (u_i + v_i)^2\right]^{1/2} \leqq U^{1/2} + V^{1/2}.$

THEOREM 8.4 (H, d) is a metric space, where d is defined by $d(u, v) = \left[\sum_{i=1}^{\infty} (u_i - v_i)^2\right]^{1/2}$.

Proof. It is readily apparent that d satisfies all the properties of a distance function with the exception of the property that $d(a, b) \leqq d(a, c) + d(c, b)$ for a, b,

$c \in H$. Let $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots)$, $c = (c_1, c_2, \ldots)$. Set $u = a - c$, $v = c - b$ so that $u_i = a_i - c_i$, $v_i = c_i - b_i$. Then $u_i + v_i = a_i - b_i$ and Corollary 8.3 yields the desired inequality.

Let E^n be the collection of points $u = (u_1, u_2, \ldots) \in H$ such that $u_j = 0$ for $j > n$. To each point $a = (a_1, a_2, \ldots, a_n) \in R^n$ we can associate the point $h(a) = (a_1, a_2, \ldots, a_n, 0, 0, \ldots) \in E^n$. Clearly h is a one-one mapping of R^n onto the subspace E^n of H. Using $d'(a, b) = \left[\sum_{i=1}^{n} (a_i - b_i)^2\right]^{1/2}$ in R^n, $d'(a, b) = d(h(a), h(b))$. Since E^n is a metric space, (R^n, d') is a metric space and h is an isometry of (R^n, d') with $(E^n, d|E^n)$.

EXERCISES

1. Let V be a vector space with the real numbers R as scalars. A function $A: V \times V \to R$ is called a *bilinear form* if $A(\alpha a + \beta b, c) = \alpha A(a, c) + \beta A(b, c)$ and $A(a, \beta b + \gamma c) = \beta A(a, b) + \gamma A(a, c)$ for scalars α, β, and $\gamma \in R$ and vectors a, b, and $c \in V$. A bilinear form is called *positive definite* if $A(x, x) > 0$, unless x is the zero vector. Define a vector space structure on Hilbert space H and show that for $u = (u_1, u_2, \ldots)$ and $v = (v_1, v_2, \ldots) \in H$, $A(u, v) = \sum_{i=1}^{\infty} u_i v_i$ yields a positive definite bilinear form.
2. Let V be a vector space with the real numbers R as scalars. A *norm* on V is a function $N: V \to R$ such that (i) $N(v) \geqq 0$ for all $v \in V$; (ii) $N(v) = 0$ if and only if $v = 0$; (iii) $N(u + v) \leqq N(u) + N(v)$ for all $u, v \in V$; (iv) $N(\alpha v) = |\alpha| N(v)$ for all $\alpha \in R$, $v \in V$. Prove that if A is a positive definite bilinear form on V, then $N(v) = (A(v, v))^{1/2}$ defines a norm on V.
3. Let N be a norm on a vector space V as defined in the previous problem. Set $d(u, v) = N(u - v)$ for $u, v \in V$. Prove that (V, d) is a metric space. Prove that the following functions are continuous: (i) $a: V \times V \to V$ defined by $a(u, v) = u + v$; (ii) $b: V \to V$ defined by $b(v) = -v$; (iii) $c: R \times V \to V$ defined by $c(\alpha, v) = \alpha v$.

For further reading, Kaplansky, *Set Theory and Metric Spaces*, Kolmogorov and Fomin, *Elements of the Theory of Functions and Functional Analysis*, and Simmons, *Introduction to Topology and Modern Analysis* all have excellent chapters on metric spaces.