Preliminary Notes for Studying General Topological Neighborhoods with Affine Transformation, Similar Subsets and Others

D. Tony Sün

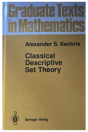
William Paterson University

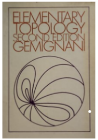
M.S. Student (f'25–) of Applied Mathematics

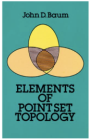
Undergraduate and Graduate Student Talks

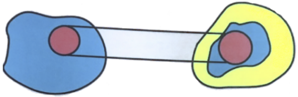
Northeastern Section of the Mathematical Association of America (NES/MAA) at Sacred Heart University, Fairfield, CT

Friday November 21 2025









(from cover of *Introduction to Topology* 3rd ed./1990 by Bert Mendelson)

A Quick Recap on Personal Learning (of Topology)

- '15–'16 academic year, a (computational) project on using the *contour trees* data structure to provide a characterizing measure to regular, digital images (domain is finite subset of $\mathbb{R}^2$)
- Fall 2016 semester: Math 456 audit (Clemson University) while in attempt of trying to prepare any mathematical context

• *Topology*: shall be the indication of, upon a given *point set* (some amount of elements), any *information* about all its open subsets

writing of pair (X, τ) "associated (info) to" the set

• That (inclusively) all subsets which are open are the definition of a topology is nonetheless 'content-specific' and subject to the particular scenario—e.g. as to how an open ball or a neighborhood will be defined, and with what metric

• Note perhaps except for sub-topology or subspace topology (as examples): with a topology presented; not clearly any 'partial topology' needed or applicable, and rather just the original entire-data-sense topology with its only associated designation of openness.

example question: is a topology (τ) "too large"?

e.g. In other words, if a subset is found to be open, but somehow it is not included the topology (or associated to any element's neighborhood), it would be an 'inaccurate situation'

..so that either the current topology needs to be "automatically adjusted" and augmented to include it, or the mechanism/definition of the (current) topology must be modified/narrowed to a more restricted version such that it *will* exclude it

Baum's Book (1962). *Elements of Point Set Topology.*

For a given point (i.e. an underlying set), its neighborhood(s) shall be an "area," or when geometric characteristics aren't obvious a collection of point(s) that "lie close to" it—as in regards to the underlying set/point set (reminder on the level of thinking). Say $X \neq \varnothing$ denotes this underlying, universal (point) set. (pp. 20-26)

Def. (Neighborhood). $x \in X$ some element then the notation $U(x)$, containing x, denotes a neighborhood of x's. Any such neighborhood shall normally be near surrounding points/elements that are "close to" x from the same set X, whilst it has yet no further requirement.

Def. (System of Neighboords / Neighborhood System, Pointwise). Let $u_x := \{U(x)\}$ denote a non-empty family/collection of "neighborhoods" of/w.r.t. each particular $x \in X$.

(1) $x \in U(x)$ for each $U(x) \in u_x$; [**notational characterization** for subset $U(x) \subseteq X$]

(2) if for some $U(x)$ there's superset $V \supseteq U(x)$, then $V \in u_x$; [supersets]

(3) if $U, V \in u_x$ then $U \cap V \in u_x$; [intersection]

(4) if $U \in u_x$, then there exists $V \in u_x$ such that if (any) $y \in V$ then $U \in u_y$; it literally infers that there exists some (other) point y we can "put" in the intersection $U \cap V$ of these two neighborhoods both to x's.

- Question (definition): what does it describe, and what might be missing (e.g. how to find it)?
- the definition 'system of neighborhoods' isn't based on a distance metric (definition)
- as properties for such nei. system; can or does it tell $\{U(x)\}$ (think strong versus weak; the 'class' of lines $\{y = 2x + b \mid \text{any } b \text{ from reals}\} \subseteq \mathbb{R}^2$ that share a slope)?

§1.1 Neighborhood Systems and Topologies (τ = an assignment of neighborhood systems for all members)
§1.2 Open Sets in a Topological Space
§1.7 Limits of Sequences; Hausdorff Spaces — notation: denote by $\{x_n \mid n = 1, 2, \cdots\}$ e.g. $(x_n)_{n\in\mathbb{N}}$ a sequence of points in the set/topological space X
§1.8 Comparison of Topologies

(pp. 22-23) usual, right order, *trivial*, and *discrete* (topologies)

Def. (**Discrete Topology**). X set non-empty then (pp. 23)

$$\tau_{\text{discrete}} := \bigcup_{x\in X} u_x := \bigcup_{x\in X} \Big\{U \mid x \in U \subseteq X\Big\}$$

is the (thus generated) *discrete topology* for X.

Def. (**Hausdorff, Topology**). For a topological space (X, τ) the topology τ is *Hausdorff* (or called "T_2 topology") provided that for each pair of distinct elements

$$a \neq b \in X$$

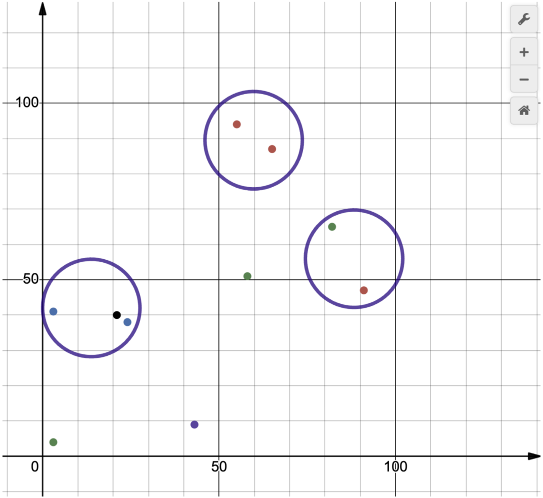
there exists two of their respective neighborhoods that don't overlap, as

$$\exists \ U \cap V \ = \ \varnothing$$

(pp. 40) where $U \in u_a$ and $V \in u_b$.

identifying neighborhoods shares much resemblance with https://en.wikipedia.org/wiki/Cluster_analysis
the *clustering problem* in data analysis ("inductive statistics")

$$X = \{(65, 87), (3, 4), (91, 47), (3, 41), (58, 51), (43, 9), (21, 40), (55, 94), (24, 38), (82, 65)\}$$



Affine/Similarity Transformation; Similarity as Relation

Def. (**Affine Transformation, Euclidean**). In the Euclidean n-space such as $\mathbb{R}^n$ $(n = 1, 2, \cdots)$ the function

$$T: \ \mathbb{R}^n \to \mathbb{R}^n$$

given by $T(x) = \lambda Rx + b$ with $\lambda \in \mathbb{R}^+$ the scaling factor, $R \in \mathbb{R}^{n \times n}$ the "rotator," and $b \in \mathbb{R}$ the "translator," is considered an *affine transformation* on (within) the n-dimensional Euclidean set.

e.g. expressing reflection w.r.t. the x-axis $(y = 0)$ in the plane

How does a pair/two elements following and under an affine transf. "correspond with" topological spaces?

The complex number tool (correspondence to $\mathbb{R}^2$) for modeling some 'fractals,' that in 2-D

$$s \cdot \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

rotates the point $(2,1)$ in column form counterclockwise by $\pi/6$ (w.r.t. the origin), scales it by $s \geqslant 0$ (w.r.t. the origin), and shifts it upward by 1 unit. This is equivalently modeled with complex numbers by

$$\begin{aligned} T: \ & \mathbb{C} \to \mathbb{C} \\ z \ \mapsto \ & (s) \, e^{i\frac{\pi}{6}} \, z + i \end{aligned}$$

Def. (Similarity Relation; Similar Subsets). X set non-empty. Define symbolically a relation $Sim \subseteq X \times X$ writing

$$X' \ Sim \ X''$$

for underneath $\left(X', X''\right) \in Sim$ meaning that "X' is similar to X''," where two subsets $X', X'' \subseteq X$ are considered.

Prop.: Any non-empty set is similar (and equal) to itself, and Sim is reflexive (default).

Limit Point for a Benchmark

(a.k.a. *accumulation point*) those that aren't a limit point are **isolated** points (real analysis)

Def. (**Limit Point**). For a subset in the reals $A \subseteq \mathbb{R}$, we say $x \in \mathbb{R}$ (not necessarily in the set) is a *limit point* if every ϵ-neighborhood, the symmetrical open interval $(x - \epsilon, x + \epsilon)$, intersects A at some point(s) other than x, namely ('deleted neighborhood')

$$\left(V_\epsilon(x) \bigcap A\right) \setminus \{x\} \neq \varnothing$$

—why exclude x here (case of an isolated element; being sequential, approaching pattern)

$$\Leftrightarrow$$

exists some sequence contained in A (values of any of the terms no beyond A) say $(a_n)_{n\in\mathbb{N}}$ for which $x \neq a_n$, $\forall n$ (**terms never touch the limit pt**.) but then $(a_n) \to x$ converges to $x \in \mathbb{R}$.

Def. (**Limit Point**, Baum's). With a topological space $(X,)$ broadly, consider non-empty subset of interest $A \subseteq X$. A point $x \in X$ (not in A necessarily) is a limit point of A's provided that for each/any neighborhood $U \in u_x$ of x's, the intersection

$$U \cap A$$

will contain some point(s) $y \neq x$.

(*What makes a sequence sequence, regarding this in the general set?*)

- *Bases* in linear algebra: important, enables coordinates; work with $\mathbb{R}^n$ instead of an "abstract vector space" (with nonetheless axioms).

- Inner product spaces: study only for real and complex (vector) spaces, i.e. a real/complex vector space associated with an inner product (e.g. $||\cdot||$); the theory of such is developed only for such $\mathbb{F}^n$ and the results usually do **not generalize to spaces over arbitrary fields**.

(Treil, 2024, Ch. 5)

Discrete Metric, and Discrete Topo. in Baum's (Example)

For X non-empty set, define function $X \times X \to \{0,1\}$ by

$$d_d(x,y) \ = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{otherwise (if } x \neq y) \end{cases}$$

as the *discrete metric* (distance) from any other point including a point itself.

Meanwhile (earlier), $\left\{ U \;\middle|\; x \in U \subseteq X \right\}_{x \in X}$

the topology tells "of whichever configuration the element is contained by a (sub)set, that is its neighborhood" that merely re-iterates property-(1) in the def. of neighborhoods (and system of neighborhoods)—obviously the entire set X is part of topology however the **empty set isn't included** for precisely all the **neighborhood**(s) are considered here.

(vs. commonly known properties of open subsets)

- from neighborhood systems to topology
- neighborhood contains the element (of interest) while being irrelevant of any topological space (a prior) nor metrics

Conclusion: Non-Metrizable Spaces

Metric-based: inducing (natural) topology always characterized by open balls (which also will need to be within each open subset)

Def. (**Metrizable Topological Space**). We say topologcial space (X, τ) is *metrizable* if τ (the topology) is induced by some metric.

Lemma: A topological space that is metrizable is also Hausdorff.

$$\text{metrizable} \;\Rightarrow\; \text{Hausdorff}$$

e.g.
- Sierpiński space, for a set $\{a, b\}$ and topology $\{\varnothing, \{a\}, \{a, b\}\}$
- real numbers with the *countable complementary* topology ($\mathbb{R}_{\text{cc}}$)
- reals with *Zariski topology*, $\tau_Z = \{\varnothing\} \bigcup \left\{\mathbb{R} \backslash S \;\middle|\; \text{where subset } S \subseteq \mathbb{R} \text{ is finite}\right\}$

Besides, the "set-level" nature of **openness** compared to element-wise characterization of the neighborhoods (a neighborhood is already an 'area' a collection of close-by points) is worth of a reminder: even though a singleton (set) can be potentially open as well. How to emphasize if necessary such difference while approaching and studying topologies in general?

With some open subsets, finite intersection or arbitrary union would "preserve openness" at the outcome of calculation. e.g. an affine-transform pair of two members from the same set (while not always trivial from subset to subset)

Reference Article: Types of Topologies (Bombelli, n.d. Ole Miss)

Types of Topological Spaces

In General > s.a. topology.
* Discrete and indiscrete topology: The discrete topology on a set X is the one in which every subset is open; They can be defined on any set.
* Finite topologies: On 3 points, there are 29; On 4 points, 355; ...; On 14 points, > 10^{23} connected ones; In general, finite topologies are the and every finite topology is generated by a partial pseudometric; Finite T_0 spaces are the same as ordered spaces.
@ Finite topologies: Kleitman & Rothschild PAMS(70) [enumeration]; Güldürdek & Richmond Ord(05) [generating pseudometric]; Barmak simplicial complexes]; Ragnarsson & Tenner JCTA(10) [smallest number of points in a topology with k open sets].
@ On discrete sets: Elizalde JMP(87); Hammer in(04)ht/98 [paths, entropy]; > s.a. Geometric Topology.
@ On closed subsets of a compact set: Sorkin & Woolgar CQG(96)gq/95 [Vietoris topology].

Regular Spaces
$ Def: The space X is regular if any $x \in X$ and $C \subset X$, with C closed and x not in C, have disjoint neighborhoods.
* Result: X is regular if the neighborhood filter of each point has a base consisting of closed sets.

Sequential Spaces
$ Def: The space (X, T) is sequential if for every open set $A \subset X$ every sequence convergent to a point in A is eventually in A.
> Online resources: see Wikipedia page.

First-Countable Spaces
$ Def: The space (X, T) is first countable if each point has a countable base of neighborhoods, i.e., for each point p in X there is a countable c each neighborhood of p contains at least one of them,

$$\text{for all } p \in X : \exists \{O_n \mid n \in \mathbb{N}, O_n \in T \text{ for all } n\}, \text{ such that for all } U, p \in U \in T : \exists\, O_n \subset U .$$

Second-Countable Spaces
$ Def: (X, T) is second countable if it has a countable base.
* Relationships: Every second countable space is first countable, Lindelöf, paracompact.
* Operations: The property is stable under taking a subspace, Cartesian product, countable union.
* Example: $\mathbb{R}^n$, with the open balls of rational radius and center, or all rational rectangles, as open sets.
> Online resources: see Wikipedia page.

T-Spaces / Separation Axioms (The terminology comes from "Trennungsaxiom") > s.a. discrete spacetimes; lines [space of causal lines]
* T_0 space: Any two distinct points have distinct sets of neighborhoods; Finite ones are in 1-1 correspondence with finite posets.
* T_1 space: For any $x \neq y$, each has a neighborhood not containing the other; Equivalently, all finite subsets are closed.
* T_2 space: See Hausdorff below.
* T_3 space: A regular T_1 space.
* T_4 space: A normal T_1 space; Every T_4 space is T_3; > s.a. Bicompact Space.
* Tychonoff space: A completely regular T_1 space.
@ References: Ali-Akbari et al T&A(09) [continuous-poset models of T_1 spaces]; Erné T&A(11) [algebraic models for T_1 spaces].

References I

"Fermat's Principle" (Wikipedia, https://en.wikipedia.org/wiki/Fermat%27s_principle)

Fowler, Michael (n.d.). "3. Fermat's Principle of Least Time." *Graduate Classical Mechanics* (Lecture Notes, UVa.). Retrieved from https://galileoandeinstein.phys.virginia.edu/7010/CM_03_FermatLeastTime.html

"Are there any non-distance based clustering algorithms?" Questions on Stats.StackExchange.com (https://stats.stackexchange.com/q/130805/495844)

Pedrotti, F. L., Pedrotti, L. M., & Pedrotti, L. S. (2017). *Introduction to Optics.* Cambridge University Press.

Skuja, A. and Hall, C. (Fall 2014). *PHYS 375: Experimental Physics III–Electromagnetic Waves, Optics and Modern Physics.* University of Maryland (Department of Physics). Retrieved from https://physics.umd.edu/courses/Phys375/Hall-Fall-2014/Physics375_Syllabus_fall2014.pdf

Treil, S. (2024). *Linear Algebra Done Wrong* (Creative Commons License). Retrieved from https://www.math.brown.edu/streil/papers/LADW

Bombelli, L. (n.d.). "Types of Topological Spaces" course notes at University of Mississippi (Physics and Astronomy).

References II

Nogin, M. (Spring 2017). "Topological Spaces (Lecture Notes 13)." *Math 110 Symbolic Logic*, California State University, Fresno.

Badzioch, B. (Fall 2018). Open Sets. *MTH 427/527 Introduction to Topology I.* SUNY Buffalo.

Lazcano, P. (2021). "Metrizable Spaces" (course notes). *MATH GU4051 Topology*, Columbia Univ.

Thank You

Fermat's "Principle of Least Time"

> *Any 'beam of light' that travels from one point to another in space will actually take a path that **would've cost the least amount of time** to travel with.*

although it appears to take an imposed assumption that light **could**'ve gone or "attempted" on multiple possible ways to get from the (same) start point to the (same) destination point, which in itself may not be a physically-experimentable process, the idea of such "optimizing over" some feasible set of/domain of options, which is nonetheless hypothetical, can be referred to for other topics about space and geometry.

But perhaps also the direct conclusion of this principle itself (which is much more vague to interpret altogether than in details).

- Imposed, not experimentable?—that any light beam that travels should travel only once, and hence no comparison is reasonable in this sense.
- Within or without mathematical assumption(s), to the view and understanding of neighborhoods

Feynman explains what Fermat did and gives the analytical solution, e.g. (`https://www.feynmanlectures.caltech.edu/I_26.html`)

"Lifeguard Problem" from *Gateway to Business Analytics with Microsoft Excel* (`https://www.depauw.edu/learn/econexcel/busanalytics/book/Tex_files/2ALifeguardProblem.pdf`)

"Oeuvres de Fermat" (1662), Internet Archive at (`https://archive.org/details/oeuvresdefermat02ferm/page/466/mode/2up`)

Initial Abstract: The absence of distance does not prevent a plausible definition of neighborhoods in a topology, and in this presentation I continue from after the summer a specific discussion on topological neighborhoods that actually provide the topology via open subsets, from a few common and intuitive examples and other tentative angles such as Fermat's Principle of Least Time. For instance if another point is affine-transformed from the point of interest perhaps with a restricted and small translator, it can be a candidate of its neighbor. Tentatively I provide a summary of non-metrizable spaces. The attempt is for a further defininition or re-definition of topological openness (e.g. $\mathbb{R}^n$ and $\mathbb{C}^n$) which is set-wise whereas neighborhood is element-wise while in the mindful advoidance of circular reasoning.

Appendix (cont.)

(photo taken by myself at William Paterson's library October 2025)