Another Preliminary Lens into Neighborhoods and Open Subsets

—with transition relating to fractals

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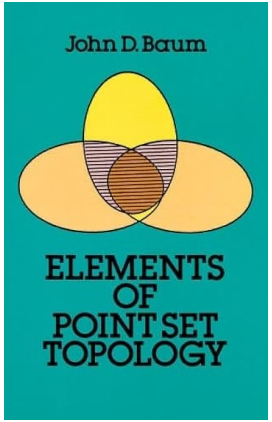
From Initial Abstract: Open (sub)sets imply metric(s) normally in the context of metric-based topology, for using the distance quantity to define open ε-balls. It is here inquired about the meaning of being a neighborhood (ε-neighborhood) associated to an element with or without a topological space.

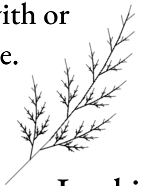
upon reading Baum's (1964): the "system of neighborhoods" has no basis of a distance/metric mentioned

- followed by a definition of topological spaces
- a neighborhood of a point is those, also as points, which "lie close to" it, all in regards to the same point set

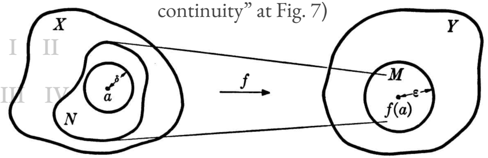
Agenda:

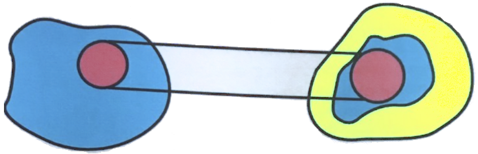
- overview of context described as of my questions.. open subsets and metrics;
- about open subsets, and topology and neighborhoods;
- on openness versus fractals;
- ~~(TBD) 1-dimensional examples~~





(depiction of second part of Theorem 4.6 "point-wise continuity" at Fig. 7)





(from cover of *Introduction to Topology* 3rd ed./1990 by Bert Mendelson)

Looking at Baum's formulation might feel a bit vague at a glance at the beginning, but it weakly inspired me to think about defining neighbors without a metric.

My inquiry begins with: what are a neighborhood, "epsilon neighborhood" or "neighboring elements" with respect to a particular element, with/only with the set; ask the same question with an associated topological space. So then to think again that *how* a particular associated topology to a set is constructed...

Brief Recap: the Context about Continuity, ε-Neighborhoods

Open (sub)sets: are 'induced' as a definition, from/with the leverage of a metric space
—which may make it not being intrinsic (e.g. compared with coordinates in Euclidean spaces)

Can (or should) the inquirer attempt defining what is a single neighbor or 'neighboring element' as a starter itself from any intuitive angles with respective a given element, such that such neighbors collectively form a neighborhood? Discussions upon what data means in this context and assumptions from topology are essential to this preliminary study.

1-Dim. Domain, to a Real-Valued/Scalar (also 1-Dim.) Codomain/Measure

Continuity 1-D Functions

We will say function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous* at $x = a \in D$ if (point-wise)

$$\forall \epsilon > 0 \text{ radius in codomain} \quad \exists \delta > 0 \text{ radius in domain, such that}$$

if x is such that (including x=a)

$$|x - a| \le \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

domain-side "a ± δ"

f(a)

i.e. given any small (and *symmetrical*) '**possible variations**' about the function value at $x = a$, these function values' pre-image (inversely-mapped points in the domain) would, if any, be "guaranteed inside" an open (and also *symmetrical*) interval—the symmetry here corresponds to the term 'radius' as well as higher dimensions.

Before Openness: Neighborhood Systems (Baum, 1964, pages 20-26)

X the universal (point) set.

Def. (Neighborhood). $x \in X$ some element then the notation $U(x)$, containing x, denotes a neighborhood of x's. Any such neighborhood shall normally be near surrounding points/elements that are "close to" x from the same set X, whilst it has yet no further requirement.

Def. (System of Neighboords / Neighborhood System, Pointwise). Let $u_x := \{U(x)\}$ denote a non-empty family/collection of "neighborhoods" of/w.r.t. each particular $x \in X$.

(1) $x \in U(x)$ for each $U(x) \in u_x$; [**notational characterization** for subset $U(x) \subseteq X$]

(2) if for some $U(x)$ there's superset $V \supseteq U(x)$, then $V \in u_x$; [supersets]

(3) if $U, V \in u_x$ then $U \cap V \in u_x$; [intersection]

(4) if $U \in u_x$, then there exists $V \in u_x$ such that if (any) $y \in V$ then $U \in u_y$; it literally infers that there exists some (other) point y we can "put" in the intersection $U \cap V$ of these two neighborhoods both to x's.

Notes and Comments: this phrasing of definition does not include/depend on an "initial generator" of what (some of) the constituents of u_x, the system, are; "any container could be a neighborhood"
in other words, it is a definition of properties for what are in the system/collection/~~span~~ of neighborhoods, here by generic notation $U(x)$. there's no "basis" yet here

Open (Sub)Sets: by definition itself — e.g., absorbs an open ball for any constituent point

e.g., of $\mathbb{R}$ the natural topology τ_{nat} is viewed as a particular 'privileged' family of open subsets

Topology: by and out of axioms/properties — i.e., the definition may seem more implicit/indirect (what is being similar)

$x \in X$ non-empty

"$U(x)$" contains x ($x \in U(x) \subseteq X$) neighborhood

$\mathcal{U}_x = \{U(x)\}$ the (?) system of neighborhoods

$\tau = \{\mathcal{U}_x \mid x \in X\}$ an "assignment" of nei. systems for each $x \in X$, as a topology which writes (X, τ)

$O \subseteq (X, \tau)$ is open if it is a neighbor to all points it contains

$\left(\begin{array}{l} \forall p \in O \\ O \in \mathcal{U}_p \end{array}\right)$

thus here **neighborhoods give/produce openness**

(note: this is not as opposed to closedness albeit usage of the word 'open')

Neighborhood system isn't unique (but by the properties satisfied accordingly);

a "particular" assignment, i.e. the specifying of nei. systems for each of all elements ⇒ a (particular) topology;

the open set definition in the last step (to the left) calls for topo. space but practically **refers to the (particular) nei. systems**

Outlining of Content: General Question(s) on Fractals vs. Openness

Fractals are the "sets with non-integral (2.6) Hausdorff dimension" or "boundary between two regions or divisions" in the Mendelbrot Set (a specific perspective); and, with competing/conflicting definitions;

"*if it combines the following characteristics:*
(a) its parts have the ***same form or structure as the whole****, except that they are <u>at a different scale</u> and may be <u>slightly deformed</u>; (b) its form is extremely irregular, or extremely interrupted or fragmented, and remains so, whatever the scale of examination; (c) it contains 'distinct elements' whose scales are very varied and cover a large range.*"

(Mandelbrot, B., 1989, "Les Objets Fractales," pp. 154)

Meanwhile, it is summarized that fractals are generally characterized by at least the following attributes (Nelson, n.d.) that every fractal

(a) is a complex structure at any level of magnification;
(b) has "non-integer dimension";
(c) has an infinite-length perimeter, but an **limited area**;
(d) is self-similar (with respect to subsets) and is "independent of scale."

fractals — of being similar to itself – "recursive patterns.."

- by "self-similarity," a fractal object can "iterate infinitely" and still land at a smaller version of the same shape;
- ponder simply about any correspondence between <u>the fractals-side **similarity**</u> (e.g., based on transformations) and <u>the topology-side of **openness**</u> (based on looking into one's subsets, etc.)?
- from the (above) correspondence and if begins with simpler examples, are these two sides of definitions equivalent somehow, if only locally to one connected subset/object?
- can examine 1-d. and 2-d. examples (1-manifolds..)

(appendix of some more pages of notes from preparation)

Main Agenda

(*transition slide*)

Plan:

→ What does it mean to **be similar** (especially when it is not synonymous to any "topological equivalence")?

→ "similarity transformation" (may be multiple definitions)

→ **similarity; and, define 'set similarity' or multiple/two sets being 'mutually similar' as a relation (reflexive, etc.) with a clear definition**

→ define fractals (once); what is meaning of the term "fractals" (in plural)?

→ *u_x* each nei. sys. is not unique;

what if it is and then how can it be conditioned for uniqueness (for each point x)?

→ assuming **different neighborhoods associated to the same point are similar** ⇒ can (mutually) similar sets form a topology---either as building blocks, generator/basic, or entirety?

→ if there's time—look at properties of (certain) unions and intersections of open sets, as well as mutually similar sets in parallel.

(quoting from 2pm Wed. talk) In nature, fractals are everywhere; the earliest mention of fractals could be Zeno's Paradox "Achilles and the tortoise" (c. 490-430 BC) 1, 1/2, 1/4, 1/8, ... as a weakly self-similar set; it is a **zero-dimensional fractal.**

Example Starter: Similarity Transformation (Affine)

Def. (Similarity Transformation). $\mathbb{R}^d$ Euclidean ($d = 1, 2, 3 \cdots$). The function

$$T: \ \mathbb{R}^d \to \mathbb{R}^d$$

given by

$$T(x) \ = \ \lambda R x \ + \ b$$

where λ, R, b are scalar, rotator, and translator respectively

$$b \in \mathbb{R}^d, \quad R \in \mathbb{R}^{d \times d} \text{ rotation matrix}, \quad \lambda \geqslant 0 \ \text{ nonnegative real.}$$

(e.g., reflection with respect to the x-axis $y = 0$ in plane)

The complex number tool (correspondence to $\mathbb{R}^2$) for modeling some 'fractals,' that in 2-D

$$s \cdot \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

rotates the point $(2, 1)$ in column form counterclockwise by $\pi/6$ (w.r.t. the origin), scales it by $s \geqslant 0$ (w.r.t. the origin), and shifts it upward by 1 unit. This is equivalently modeled with complex numbers by

$$T: \ \mathbb{C} \to \mathbb{C}$$
$$z \ \mapsto \ (s)\, e^{i\frac{\pi}{6}} \, z + i$$

Similarity

Def. (Similarity Relation; Similar Subsets). X set non-empty. Define symbolically a relation $Sim \subseteq X \times X$ writing

$$X' \; Sim \; X''$$

for underneath $\left(X', X''\right) \in Sim$ meaning that "X' is similar to X''," where two subsets $X', X'' \subseteq X$ are considered.

Prop.: Any non-empty set is similar (and equal) to itself, and Sim is reflexive (default).

Using this, the specific conditions/definitions for when the similarity holds can be various, such as using quantitative requirements for topological characteristics.

For instance, the existence of a *similarity transformation* (mentioned) can give a specific similarity (relation) definition—as when the inverse transform is also available ⇒ similarity is ***symmetric***

Fractals (as appeared in plural) can be understood as repeated/repeating **paterns**—and for which a pattern shall exist as its own definition (e.g., a periodic function) whilst with any/some mathematical objects (e.g., Sierpinski gasket SG_2 as a graph as vertex/edge lists).

A particular study/research question about fractals will, indeed, be based upon some fractals as the context, whereas not necessarily about geomegric/topological characteristics; compared to a topic like differentiable manifolds (manifolds are always involved) and algebraic topology (topological questions throughout). It would depend on the direction/aspect of the specific project, such as about harmonic functions (on a particular metric), about relationships with natural numbers, etc.

Step(s) of Fractals in Corresponding to the Properties of Topology

Prop. Fractal/fractal collection $\{F_n\}_{n=0,1,2\cdots N-1} = \mathcal{O}$ (tentative notation) with a finite number of N elements/constituent objects in a sequence. Here take the assumption of $F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{N-2} \subsetneq F_{N-1}$ with F_{N-1} being the "biggest" superset (contains every element).

F_{N-1} —— X

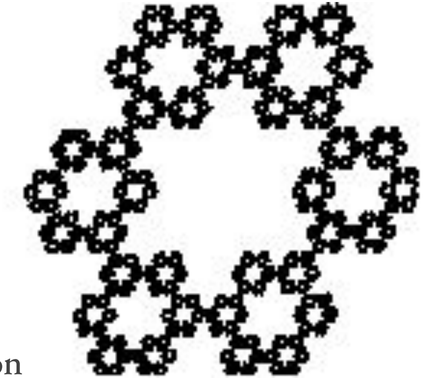
$\{F_n\}$ —?— τ

(i) an arbitrary union, of any number of elements from $\mathcal{O}$, is in $\mathcal{O}$;

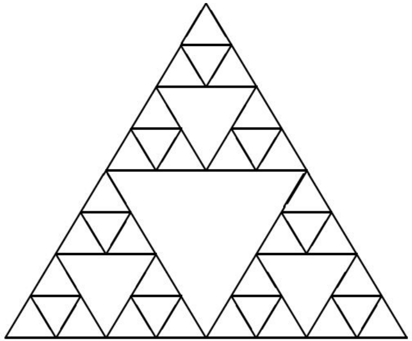
(ii) a finite intersection of any elements from $\mathcal{O}$ is in $\mathcal{O}$;

Justification is that indeed making a union yields the largest set, and an intersection the smallest (due to successive containment).

Question: What if when F_{N-1} the particular say largest fractal is considered in its actual form (rather than in parallel to other fractal-object elements)?



Sierpinski hexagon



$SG_2(3)$

Conclusion

Different patterns lead to (correspond to) different branches of mathematics:

Arithmetic

Number Theory

Geometry

Calculus

Logic

Probability Theory

Topology

Fractal Geometry

Since the Nineteenth Century the major change in emphasis in math “wasn’t arbitrary… …[the process] came about through the increasing complexity of the world…procedures and computation will not go away—they’re still important—but in today’s world they’re not enough, and we need *understanding*.

They (patterns) can be either real or imagined patterns, visual or mental, static or dynamic, qualitative or quantitative, utilitarian or recreational, from pursuits of science of from the *inner workings of the human mind*.”

Thank You

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Conclusion

"..the (crystal element technique) can turn anything it touches into crystal (crystalize), in other words, water, earth or wood elements won't have any effect.."

Kobayashi, O. [Director]. (Mar. 2011). Painful Decision. *Naruto Shippuden* (episode 201). Pierrot Co., Ltd. Available at Crunchyroll at http://www.crunchyroll.com/watch/G6JQ2K9ER/painful-decision

Briefly Restating the Context about Continuity, ε-Neighborhoods, Topo. Spaces (cont.)

Def. 4.4 (**Neighborhood**, and **Complete System of Neighborhoods**). Let (X, d) be a metric space, and $a \in X$. A subset of this space $N \subset X$ is called a *neighborhood* of element a if there exists some $\delta > 0$ such that

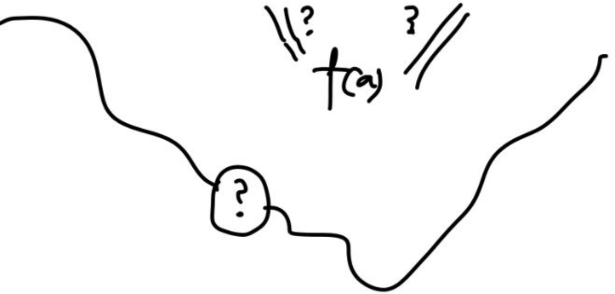
(goes to a real, $X \times X \to \mathbb{R}$)

$$\underset{\text{center}}{B(a;}\ \underset{\text{radius}}{\delta)} \subseteq N \qquad \text{definition} := \{x \in X \mid d(a, x) < \delta\}$$

that is this subset contains some (possibly arbitrarily small) "open ball" "centered at $a \in X$.
Hence, bringing all possible such neighborhoods of a's into a collection $\mathcal{N}_a$ we get the *complete system of neighborhoods* of element/point $a \in X$.

$(a+x=a)$

eg left limit = right limit



Theorem 4.6 (**Continuity by "Enclosed Neighborhood"**). Consider function defined from one metric space to another, denoted

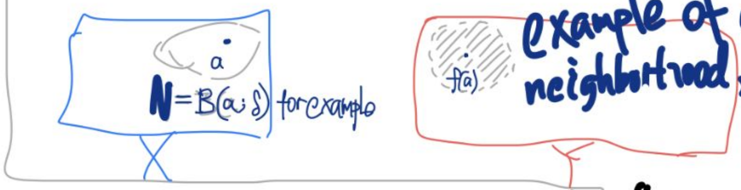
more about domain $\quad f : (X, d) \to (Y, d')$ $\quad$ f(a) Y

then we say f is continuous at $a \in \boxed{X}$ *iff* **for each/any neighborhood** M of $f(a)$ in co-domain, there exists a corresponding domain-side neighborhood N of a such that

$$f(N) \subset M, \quad \text{as well as (equivalently)} \quad N \subset f^{-1}(M) \quad \text{pre-img.}$$

that is to say the existence of an, "any" enclosed ~~forward-imaged~~ neighborhood defines continuity. open

~ "ε-neighborhood" that is a (particular) ball under d



, M but this depends on what d' actually is;

'System' of Neighb asks for traversing/varying over sizes of radius (δ)

"Spatially adjacent" Not enough

Why Continuity (any leverage)? — it demands a "synchronicity" between two metric functions

$$\underset{\subseteq \text{codom}}{\forall M} \overset{\text{depends on}}{\sim\!\sim} d' \cdots\cdots \underset{\subseteq \text{dom.}}{\exists N} \sim d$$

But openness (itself) is about just one set ..... "X" when set aside metrics

(rather "clustering")

Not as opp. to being open

Notion of "Closeness" for Points

... say which (≥ 2) points "lie close" to either each other or some other reference object / point

"In a certain sense, a neighborhood of a point x is the set of points which lie 'close' to the point." (Baum, 1964)

Def (Neighb. System) X set For each point $x \in X$, let $\mathcal{U}_x = \{U(x)\}$ be a non-empty family of subsets of X associated w/ x, ...

next page

Def. (Sys. of Neighboords, Pointwise). ..... such that

(Baum's book)

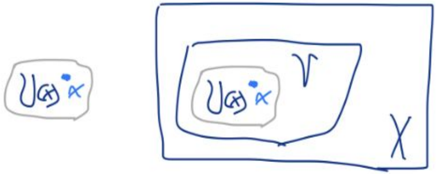
the "universal" set that corresponds to what the topo. space is constituted of - - - X any set; $x \in X$ any element

$$u_x := \{U(x)\}$$

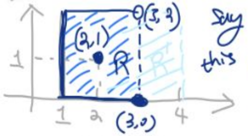
(1) $x \in U(x)$ for each $U(x) \in u_x$; [**notational characterization** for subset $U(x) \subseteq X$]

$\subseteq X$

(2) if for some $U(x)$ there's superset $V \supseteq U(x)$, then $V \in u_x$; also by definition [supersets] directly from #(1)



e.g.

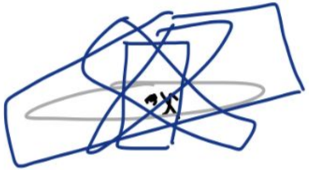


say this $R \subsetneq \mathbb{R}^2$ is neighborhood to $x^* = (2,1)$; $R \in \underbrace{\mathcal{U}_{x^*}}_{} = \{$any neighborhoods of $x^*\}$ then this superset R' (bigger enclosing rect) is also a neighborhood

(3) if $\underline{U, V \in u_x}$ then $U \cap V \in u_x$; [intersection]

notation ⇓

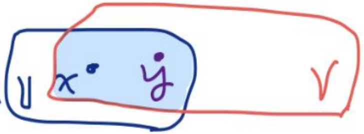
$x \in U$, $x \in V$ $\Rightarrow x \in (U \cap V) \in \mathcal{U}_x$



intersections of elmt's in the system $\mathcal{U}_x$ stays in the system (closure)

(4) if $U \in u_x$, then there exists $V \in u_x$ such that if (any) $\underline{y \in V}$ then $U \in u_y$.

can "put" some point y inside this intersection



In general, $U \subseteq X$ (the univ. set), $V \subseteq X$ (hypothetically); know that $x \in U \subseteq X$ namely $\boxed{U \supseteq \{x\}}$ (singleton set); similarly $x \in V$, too w/ $y \in V$ (assumption) What if $x = y$? Otherwise? (also $V \neq \emptyset$ noted)

Notes and Comments: this phrasing of definition does not include/depend on an "initial generator" of what (some of) the constituents of u_x, the system, are; "any container could be a neighborhood"

in other words, it is a definition of properties for what are in the system/collection/~~span~~ of neighborhoods, here by generic notation $U(x)$. there's no "basis" yet here

Metric

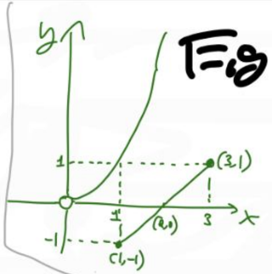
Euclidean $X \subseteq \mathbb{R}^d$ ...... can give a corresponding topology

"$(X, d) \rightsquigarrow (X, \tau_d)$"

metric space — topo. space

Note: A topological space is NOT always a metric space

(i.e., there are topological spaces in which the given topology cannot come from a metric.)



E.g. $\{(x,y)\in\mathbb{R}^2 \mid x>0 \text{ AND } y=x^2\} = \{(x,x^2) \mid x\in\mathbb{R}^+\} = X_1$

$M = \frac{1-(-1)}{3-1} = \frac{2}{2} = 1;$

$y-1 = (1)(x-3)$

$y = x-2$

$\{(x, x-2) \mid x\in\mathbb{R},\ 1\le x\le 3\} \equiv X_2$

$X = X_1 \cup X_2 \subsetneq \mathbb{R}^2$

$d: \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ specified by

$(x_1,y_1),(x_2,y_2) \mapsto |x_1-x_2| + 2\cdot|y_1-y_2|$

$\Rightarrow (X,d)$ met. space; then use d to define ϵ-neighborhood / open ball, for "τ_d"

("richest" structure)

X "inner product space" $\xrightarrow{\text{parallelogram law}}$ X normed space $\rightsquigarrow$ metric space X, d $\xrightarrow{\text{make/generate}}$ topological space X, τ_d

"upgrade"

a "special kind of topology"

$x\in\mathbb{R}^n,\ y\in\mathbb{R}^n$

$\langle x,y\rangle \equiv x_1y_1 + \cdots + x_ny_n$

e.g., $\|x\| = \sqrt{\langle x,x\rangle}$

$= \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$

$d(x,y) \equiv \underbrace{\|x-y\|}_{L_2\text{-norm distance}}$

only makes sense for Euclidean n-space

OR, two functions f and g

$\langle f,g\rangle \equiv \int_a^b f(x)g(x)\,dx$

(Rudin's book, chapter 1 ?)

Riemann Integral $\xrightarrow{\text{generalize}}$ Lebesgue Integral — "measure" on domain side — "Lebesgue-measurable"

If element $x^* = (3,9) \in X_1 \subseteq X \subseteq \mathbb{R}^2$, then say $\delta>0$ is some small radius, open ball $B((3,9);\delta)$ should be in any neighborhood of $(3,9)$;

Metric (cont.)

The **absence of distance** when talking about neighborhoods might be **intentional**, because we don't need a distance function to define any/a particular neighborhood. For example, the book *Topology* by James R. Munkres says a neighborhood of a point *x* is any open set containing this point; Munkres also points out that other authors simply use the term "neighborhood of a point" to indicate likewise—a neighborhood to *x* is any set that contains an open set *U*, such that *x* is an element of *U*. In either case, there is yet no reference to distance (but, there is a reference to a topology).

Baum's book—"system of neighborhoods" definition may expect to motivate an upcoming, more topological definition for neighborhood(s). This could explain why it comes before the definition of a topology. Regardless, the absence of a distance was important & worth noticing to me, for it keeps everything in a general setting—not all spaces need a distance function, and studying each doesn't always require a metric.

(*my notes combined with discussion with some classmate during last Fall '24 semester, at CUNY's Graduate Center*)

It then follows (my) two immediate questions: (i) when I see/read the word "open set" first I recall it is a "subset," and it needs to reside in an underlying topo. space (and that requires a metric/metric space);

→ "I'm not sure how your book defines it, but, typically, a topological space doesn't require a metric space. The relationship between the two is that a metric space is a specific topological space. The metric on the space is used to define open sets and **these open sets are used to form the topology**. Every metric space is a topological space, but not every topological space is a metric space."

Metric (cont.)

(ii) doesn't Cartesian product (itself) already imply being Euclidean? —if the answer is yes, then the *n*-space (e.g., *xy*-plane, *xyz* 3-space) is already "rigid," meaning each point's coordinate exists in the first place, then any other metric defined based upon it would seem more like a re-definition of distance to me..

→ "The Cartesian product does not imply being Euclidean. This entirely depends on **whether or not an Euclidean distance can be put [applied] on the space.** Not every space has a Euclidean distance on it because some spaces don't even have a metric." And, some space might not have ordinary coordinates/coord. maps.

My interpretation by far is that a metric is actually "utilized" to define open sets, which induces the formation of topology (i.e., I have a topo. space then as a result). Still, even if some metric had already been chosen, I wonder if there should/needs to be a metric space there, in the first place...

Regarding open balls—it can be viewed as like a door/access to "augment the space" with at/from a very small locality, a few points, or a single point...

39th Summer Topology and Its Applications Conference (**General and Set-Theoretic Topology**)

UNIVERSITY OF SOUTH ALABAMA

ScholarLattice (Aug. 2025)

"Intrinsic Retraction for General Topological Neighborhoods" — D. Tony Sün

Topological equivalence among subset objects within the same space is of core algorithmic and applied interests. An early inquiry for topological spaces has...

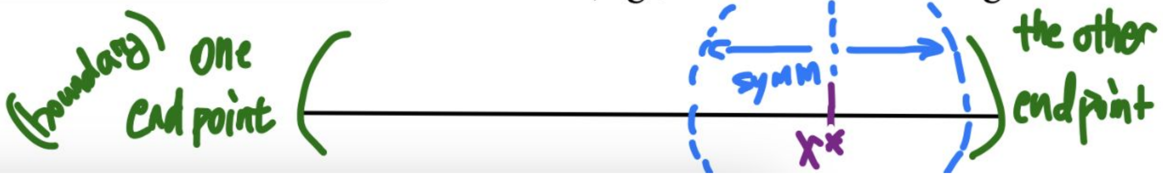
Context about Continuity, ε-Neighborhoods, Topo. Spaces (cont.)

- What can we say about being "continuous"? What about "openness" (what, and how)?
- If for real intervals (1-d.), being able to find such open-interval subset makes it open (circular reasoning)

  comment: the characterizing of a subset "expanding from" a single point/element already uses a metric, which makes it (openness) metric-based.
- Difference between an element ("singleton") versus anything else non-empty (e.g., sets greater than 1 by count of constituent elements, unions of single elements, uncountably many elements)
- Can the process of expanding a single element, into anything bigger than itself, be one that "opens it up" (one versus many)?
- Attempts to explain/dispute that potential circular reasoning (as a concerning/vagueness part of learning)

  or by a process, an operation
- "Open" as opposed to being closed; configuration implies <u>an inward (or outward) "motion"</u> as it can be equivalently represented by <u>a change in constituents/content</u> for the set (e.g., check that of a regular 1-d. open interval, regarding this)

set that is open $\xrightarrow{\text{subset}}$ (same)



Fractals versus Openness: The General Question

Fractals are

the boundary between two regions/divisions in the Mendelbrot Set;

with competing/conflicting definitions;

"*if it combines the following characteristics:*

*(a) its parts have the **same form or structure as the whole**, except that they are at a different scale and may be slightly deformed; (b) its form is extremely irregular, or extremely interrupted or fragmented, and remains so, whatever the scale of examination; (c) it contains 'distinct elements' whose scales are very varied and cover a large range.*"

(B. Mandelbrot. "Les Objets Fractales," 1989, pp. 154)

Meanwhile, it is summarized that a fractal is characterized by at least the following attributes (Nelson, n.d.) that it

(a) is a complex structure at any level of magnification;

(b) has "non-integer dimension";

(c) has an infinite-length perimeter, but an **limited area**;

(d) is self-similar (with respect to subsets) and is "independent of scale."

fractals — of being similar to itself – "recursive" patterns..

- by "self-similarity," a fractal object can "iterate infinitely" and still land at a smaller version of the same shape;
- ponder simply about any correspondence between the fractals-side **similarity** (e.g., based on transformations) and the topology-side of **openness** (based on looking into one's subsets, etc.)?
- from the (above) correspondence and if begins with simpler examples, are these two sides of definitions equivalent somehow, if only locally to one connected subset/object?
- looking at 1-dimensional examples (1-manifolds, $\mathbb{R}$)

(out of 2)

Another Preliminary Lens into Neighborhoods and Open Subsets

June 2025

532 Malott Hall 6/16 9AM

Q12. **Please submit your Talk Abstract below. If you do not have it at this time, please submit it by March 7th. You will receive a confirmation email once you submit your registration, which provides a link to edit your registration.**

Open (sub)sets should imply some metric, because one needs a metric for distance to define open sets. Upon reading the book by Baum's the "system of neighborhoods" did not have a basis or mention for how such distance is measured, and instead a definition of topological spaces comes after this portion. A neighborhood of a point is those, also as points, which "lie close to" it, all in regards to the same point set. This formulation might be vague at a glance for undergraduate students, but it weakly inspired me to think about defining neighbors without a metric. My inquiry here is to think about and understand more in details what it means to be a neighborhood or "epsilon neighborhood," to an element in the point set with or even without a topological space pre-conceived in the first place. After all, it reads to me as, conventionally, open subsets are an "induced definition" from a metric space. Can or should the inquirer attempt defining what is a single neighbor or "neighboring element" as a starter itself from any intuitive angles with respective a given element, such that such neighbors collectively form a neighborhood? Discussions upon what data means in this context and assumptions from topology are essential to this preliminary study.

"Varieties" (Variety) was to refer to self-similarity …

P2.81 P.31

Art 12. Introduction to Fractals — from U. Nebraska course website (U. of Wollongong)

Fractals = geometric figures (rectangles, circles, squares) which have special properties that other figures "do not have," which practically differentiate fractals from non-fractals

§12.1 Mandelbrot Set & Self-Similarity — what generated by simple iterative processes; some applications; can arise in simple models of physical processes

§12.2 Making a Fractal: the Sierpinski Triangle

§12.3 Koch Snowflake

§12.4 Applications

§12.5 Modeling

Response Summary:

8th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals - 2025

June 16, 2025 - June 20, 2025

You will receive a notification that your registration has been recorded at the end of this registration survey and a confirmation e-mail when your registration is complete.

Q2. First Name

Q3. Last Name

Q4. Home Institution

Yeshiva University

Q5. What is your position?

• Grad Student

The boundary between the two regions, the "inner region" and the "outer region" divided from the 2D plane, is a fractal (as example).

A "figure" is said to be self-similar if some magnified subsets ~~of the object~~ "look like the whole and to each other," in that ~~speed~~ the subsets need not look exactly the same as each other at all scales.....

Fractals "infinite amount of detail"

(2/out of 2)

39th Summer Topology and Its Applications Conference — S. Alabama 8/11/2025

(Similar to what's submitted to Cornell?)

Can neighborhood be (already) defined by "contraction" of some kind?

S — In the 'extreme' case, for such set is a point set char. by its constituent elmts, use all elements to construct a graph object/data structure. 5/18/2025

→ might try to read about the "contraction mapping theorem" and retraction to begin

(to continue)

General and Set-Theoretic Topology — Submissions are now open, and will close on 2025-05-31 11:59PM [Central Time (US & Canada)] (by June 15th)

Talks involving set-theoretic and other techniques used to investigate topics in general topology. Organizers: Chris Caruvana, Jared Holshouser, Lynne Yengulalp.

"Set-theoretic topology" is the study of techniques from set theory that are used to investigate topics in general topology. Such topics include topological games, homogeneity, hyperspaces, topological algebra, compactifications, and much more. Problems about topological topics or properties often require extra axioms of set theory to answer. Advances in set theory, conversely, have applications in general topology. Recent work by speakers in this session include set-theoretic schemes that build uncountable structures from finite ones (Osvaldo Guzman), persistent properties in uniform box products (Jocelyn Bell), resolvability in certain classes of topological spaces (Lajos Soukup), compact generators in function spaces (Paul Gartside), cardinal inequalities and pi-character (Ivan Gotchev and Vladimir Tkachuk), zero-dimensional sigma-homogeneous spaces (Andrea Medini), and others.

topology-side, openness

fractals-side, self-similarity

(Feiberg, 2016)

Def. Sim. Transformation

For any $x \in \mathbb{R}^d$ $(d=1,2,3,\dots)$, the function $T:\mathbb{R}^d \to \mathbb{R}^d$ given by $T(x) = \lambda_i R x + b$

(next)

- What does it mean to **be similar** (especially when it is not synonymous to any "topological equivalence")?
- "similarity transformation" (may be multiple definitions)
- **self-similarity**
- define fractals (once); what is meaning of the term "fractals" (in plural)?
- *u_x* each nei. sys. is not unique; what if it is and then how can it be conditioned for uniqueness (for each point x)?

$b \in \mathbb{R}^d$ is a "translator" or translation ~~constant~~ vector term

$R \in \mathbb{R}^{d \times d}$ (square matrices) is a rotation matrix (also constant).

$\lambda \geq 0$ is a scaling factor/scalar (also constant)

e.g. "reflection" is a type of sim. transformation

(3)

$\lambda = |a| \geq 0$ is the norm of a

$\theta \in \mathbb{R}$ is the (angular) argument of

$b \in \mathbb{C}$ still a translation factor

such that $T(z)$ "dilates" a complex number/2D point by $\lambda \geq 0$, then rotated by radian angle θ, then finally translated by b.

~~2-Dim. Case ($\mathbb{R}^2$) in Terms of the Complex Plane~~

~~When a point $x \in \mathbb{R}^2$, denote $z \in \mathbb{C}$ by complex number notation, namely $z \overset{\text{denoted}}{=\!=} x + i \cdot y$ is any complex number, where $(x, y) \in \mathbb{R}^2$.~~

* Why is "$\lambda R x$" equivalent to "az" for $x \in \mathbb{R}^2$

Example of 2 dimensions: let $z \in \mathbb{C}$ denoted $z = x + iy$ and

$$T : \mathbb{C} \to \mathbb{C}$$

$$z \mapsto az + b \quad \cancel{\text{with}}$$

where

$$\begin{cases} a = \lambda e^{i\theta} \in \mathbb{C} \\ \cancel{(\lambda \geq 0)}, (\theta \in \mathbb{R} \text{ any angle}) \end{cases}$$

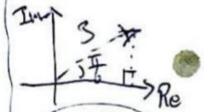
e.g.,

$(2+i) \in \mathbb{C}$

$(2,1) \in \mathbb{R}^2$

$$\begin{cases} a = 3\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \boxed{3e^{i\frac{\pi}{6}}} \in \mathbb{C} \\ b = i \in \mathbb{C} \end{cases}$$

then transform a point $z^* = 2 + i$ becomes



$3e^{i\frac{\pi}{6}}$

$= 3\left[\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right]$

$= 3\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 3$

$$T(2+i) = a \cdot (2+i) + b$$

$$= 3\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \cdot (2+i) + (i)$$

$$= 3\left(\sqrt{3} + \frac{\sqrt{3}}{2}i + i - \frac{1}{2}\right) + i$$

$$= 3\left(\sqrt{3} - \frac{1}{2}\right) + \frac{3}{2}\sqrt{3}\,i + 3i + i$$

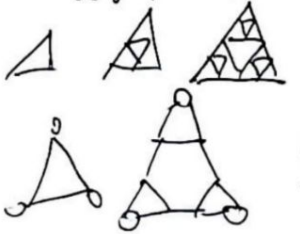
$$= 3\left(\sqrt{3} - \frac{1}{2}\right) + \left(\frac{3}{2}\sqrt{3} + 4\right)i\,;$$

- assuming **different neighborhoods associated to the same point <u>are similar</u>** ⇒ can (mutually) similar sets form a topology---either as building blocks, generator/basic, or entirety?
- How, when possible, do neighborhood system(s) form a topology (of the universal set)?
- if there's time—look at <u>properties of (certain) unions and intersections</u> of <u>open sets</u>, as well as <u>mutually similar sets</u> in parallel.
- **[move it to earlier] define <u>set similarity as a relation</u> (reflexive, etc.) w/ a clear definition**

(mentioned in 2pm Wed. talk) In nature, fractals are everywhere; the earliest mention of fractals could be Zeno's Paradox "Achilles and the tortoise" (c. 490-430 BC) 1, 1/2, 1/4, 1/8, ... as a weakly self-similar set; it is a **zero-dimensional fractal**.

(5pm Monday by N.)

a "graph sequence"



G_n

H_n

J_n

that is $2+i \mapsto 3\left(\sqrt{3}-\frac{1}{2}\right) + i\left(\frac{3}{2}\sqrt{3}+4\right)$

under T by $T(z) = \boxed{3e^{i\frac{\pi}{6}}}\, z + i$, $z \in \mathbb{C}$

"a" =

It shall be equivalent to some similar transform of point $(2,1)$ into $\left(3(\sqrt{3}-\frac{1}{2}), \frac{3}{2}\sqrt{3}+4\right)$ both in $\mathbb{R}^2$.

Try verifying with point $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ column vector,

under a rotation by ~~matrix le~~

~~left-multiplying matrix~~ $\begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$,

(4)

What it means to "get a Laplacian" of my object?

One way to understand Δ (Laplacian) is analogous to Fourier theory? "eigenfunction"

$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3}{4}$

$\begin{bmatrix} \cos\frac{\pi}{6} & 0 \\ 0 & \sin\frac{\pi}{6} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} = \frac{3}{2}$

$(1,0) \mapsto \left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right)$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{\sqrt{3}}{4} \end{bmatrix}$; $\frac{3}{4} = a$, $\frac{\sqrt{3}}{4} = c$

followed by scaling by scalar-multiply $\sqrt{3}$

and lastly translated by vector-addition $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$.

$(3) \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$

$(3) \cdot \begin{bmatrix} \sqrt{3}-\frac{1}{2} \\ 1+\frac{\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= (3\sqrt{3} - \frac{3}{2}$

~~$3 \cdot e^{i\theta}$~~

~~$3 \cdot (\cos\theta + i\sin\theta)$~~

~~$z = \sqrt{3}\, e^{i\theta}$~~

3am

Jun. 16 Monday

"Dirichlet forms" 2-dim.

"Reconstruction of "Analysis" not starting from derivation"

criterion for general metric spaces

$\frac{1}{\text{energy}}$ ~→ register? resistance

conductive homogeneity

(... of locally symmetric self-similar sets)

a class of self-similar sets, which have local symmetry ......

Regular J-gon,

$Q_J \subseteq \mathbb{R}^2$ "with its center"

$\Gamma_J = \{\varphi \mid \varphi \in O(2),$ symm. group of Q_J

"G" for the group of global symmetries of $Q_J \subseteq \mathbb{R}^2$

$J=8$, $G = \mathrm{Rot}_4 \,//\, (\mathbb{Z}_9)^e = \{1,2,4,5,7,8\}$

$$\begin{bmatrix} 3\sqrt{3} - \frac{3}{2} \\ \cancel{4\sqrt{7}} \\ \frac{3}{2}\sqrt{3} + 4 \end{bmatrix}. \ (\checkmark)$$

"Spectral decimation"?

Hence if az is the multiplication term ($a, z \in \mathbb{C}$), then it corresponds to a rotational matrix of the angle

$$\mathrm{Arg}(a) \in (-\pi, \pi]$$

the main argument of the complex number a, and also the scaling multiplier $|a| \geq 0$ the magnitude of a.

Def. (Self-similar Sets). An Euclidean $X \subseteq \mathbb{R}^d$ (for some $d = 1, 2, 3, \dots$) is said to be self-similar or as a self-similar set if

(from Monday talk)

(X, d, μ) complete metric measure space, when (X, d) is a metric space & μ a Borel regular measure

⋮

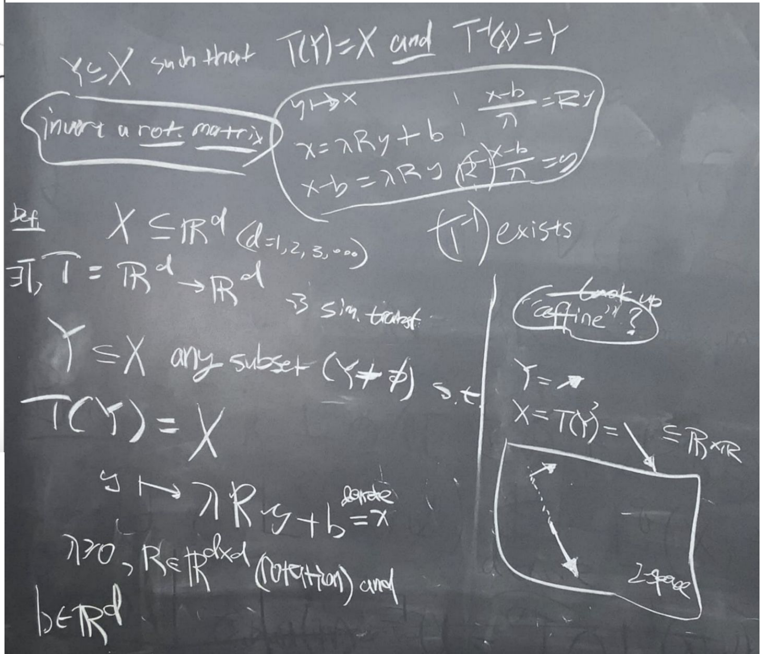
"pointwise fractional (s,p,q)-Hardy inequality"

~~"three ways of self-improvement"~~

"self-similar sets/measure"

"potential is locally bounded"

Killed Brownian motion



My note–the objects of fractals (such as SG2) w/ any form of sketching of a diagram, are not assumed to be "residing in 2D," because they are rather graphs/combinatorial objects characterized by vertices, edges, cells, etc.